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SYMMETRY GROUPS AND GROUP INVARIANT SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction

The application of the theory of local transformation groups to the study of partial differential equations has its origins in the original investigations of Sophus Lie. He demonstrated that for a given system of partial differential equations the Lie algebra of all vector fields (i.e., infinitesimal generators of local one-parameter groups transforming the independent and dependent variables) leaving the system invariant could be straightforwardly found via the solution of a large number of auxiliary partial differential equations of an elementary type, the so-called "defining equations" of the group. The rapid development of the global, abstract theory of Lie groups in the first half of this century neglected these results on differential equations for two main reasons: first the results were of an essentially local character and secondly, except for the case of ordinary differential equations, the symmetry groups did not aid in the construction of the general solution of the system under

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consideration and thus appeared to be of rather limited value. The early investigators had failed to discover the concept of a group invariant or self similar solution, and it was not until after 1940, beginning with the work of L. I. Sedov [21] and G. Birkhoff [2] on a general theory of dimensional analysis, that essentially new research into this area was begun. While Birkhoff and Sedov first considered only scale invariant solutions, it was soon realized that group invariant solutions could be found for arbitrary local transformation groups, and their construction involved the solution of partial differential equations in fewer independent variables. (The terms self-similar, symmetric and automodel solutions have all been used in the literature to describe the concept of a group invariant (or G-invariant, if G is the particular group) solution.) Finally, in the early 1960's the fundamental work of L. V. Ovsjannikov on group invariant solutions demonstrated the power and generality of these methods for the construction of explicit solutions to complicated systems of partial differential equations. While only local in nature, Ovsjannikov's results provided the theoretical framework for commencing a systematic study of the symmetry groups of the partial differential equations arising in mathematical physics. This work is being pursued by Ovsjannikov, Bluman, Cole, Ames, Holm and others, and has provided many new explicit solutions to important equations. In another direction, the recent deep investigations of Goldschmidt and Spencer [26] find general conditions for the solvability of the defining equations of a symmetry group, a question which will not be dealt with here.

The primary purpose of this paper is to provide a rigorous foundation for the theory of symmetry groups of partial differential equations, and to thereby prove the global counterparts (and counterexamples) of the local results of Ovsjannikov. This will be accomplished primarily in the language of differential geometry, utilizing a new theory of partial differential equations on arbitrary smooth manifolds, which generalizes the theory of differential equations for vector bundles. Let Z be a smooth manifold representing the independent and dependent variables in the equations under consideration; in the classical case Z will be an open subset of the Euclidean space $\mathbf{R}^p \times \mathbf{R}^q$ with coordinates $(x, u) = (x^1, \cdots, x^p, u^1, \cdots, u^q)$, where the x's are the independent and the u's the dependent variables. We shall construct a fiber bundle $J_k^*(Z, p) \to Z$ called the extended k-jet bundle, which corresponds to the various partial derivatives of the u's with respect to the x's of order $\leqslant k$. In the special case that Z is a vector bundle, the extended k-jet bundle will be the "completion" of the ordinary jet bundle $J_k Z$ in the same sense that projective space is the "completion" of affine space. In this context, a k-th order system of partial differential equations will be described by a closed subvariety Δ_0 of $J_k^*(Z, p)$. A solution of Δ_0 will be a p-dimensional submanifold of Z, called a p-section, whose extended k-jet, a p-dimensional submanifold of $J_k^*(Z, p)$ lying over the original submanifold of Z, is entirely contained in Δ_0. If G is a local group of transformations acting on Z, there will be an induced action of G on $J_k^*(Z, p)$, called the k-th prolongation of the action of G, which comes from the action of G on p-sections. The corresponding prolongation of the infinitesimal generators of G has a relatively simple expression in local coordinates, and can be used to check via the standard infinitesimal criteria whether a system of partial differential equations Δ_0 is invariant under the prolonged action of G, which implies that G transforms solutions of Δ_0 to other solutions. Here this local coordinate expression will be derived using the techniques of symmetric algebra; it does not seem to have appeared previously in the literature. It allows for a much more unified and straightforward derivation of the symmetry groups of higher order partial differential equations.

Now suppose that G acts regularly on Z in the sense of Palais [19] so that there is a natural manifold structure on the quotient space Z/G. If Δ_0 is a system of partial differential equations on Z which is invariant under G, then the problem of finding all the G-invariant solutions to Δ_0 is equivalent to solving a "reduced" system of partial differential equations $\Delta_0/G \subset J_k^*(Z/G, p - l)$, where l is the dimension of the orbits of G. In other words, the solutions of Δ_0/G are $(p - l)$-dimensional submanifolds of Z/G, which, when lifted back to Z, provide all the G-invariant solutions to the original system Δ_0. The important point is that the number of independent variables has been reduced by l, making the reduced system in some sense easier to solve. (Although this does not always hold in practice; there are examples where Δ_0 is linear, whereas Δ_0/G is a messy nonlinear equation.) It is this fact which makes the symmetry group method so useful for finding exact solutions to complicated differential equations. Note that since we must consider the quotient space Z/G, which cannot be guaranteed to possess any nice bundle properties due to the arbitrariness of the action of G on the independent and dependent variables, we are forced to develop the aforementioned theory of partial differential equations over arbitrary smooth manifolds which have no a priori way of distinguishing the independent and dependent variables. The question of when the distinction can be maintained is discussed in some detail in the author's thesis [16].

A few brief comments on the organization of the material in this paper are in order. §2 reviews the theory of symmetric algebra of vector spaces and its applications to the derivatives of smooth maps between vector spaces. The most important result is Theorem 2.2, the Faa-di-Bruno formula for the higher order differentials of the composition of maps; it forms the basis of many of the subsequent calculations, in particular providing a useful explicit matrix representation of the prolongations of the general linear group. §3 develops the concept of the extended jet bundle over an arbitrary manifold and gives both the bundle-theoretic and local coordinate descriptions of these objects in detail. In §4 we show how the concepts of differential operators and systems of partial differential equations are incorporated into the extended jet bundle theory, and discuss the concept of prolongation.

§5 applies the theoretical concepts of the preceding three sections to the prolongation of local transformation group actions. In particular, the local coordinate expression for the prolongation of a vector field is derived. In §6 we show how this formula is applied in practice to find the symmetry group of a system of partial differential equations, and illustrate this with the derivation of the symmetry group of Burgers' equation, an important evolution equation arising in nonlinear wave theory. §7 takes up the problem of equivalent systems of partial differential equations, the prototypical example being an n-th order partial differential equation and the first order system obtained by treating all the derivatives of order $< n$ as new dependent variables. It is shown that, barring the presence of "higher order symmetries," the groups of two equivalent systems are isomorphic under a suitable prolongation. An example of an equation with higher order symmetries is discussed. Finally, in §8 the fundamental theorem on group invariant solutions is proven and is applied to deriving some interesting group invariant solutions to Burgers' equation.

The results of this paper form a part of my doctoral dissertation written under Professor Garrett Birkhoff at Harvard University. I would like to express my profound gratitude to Professor Birkhoff for his constant help, suggestions and encouragement while I was writing my thesis.

2. Symmetric algebra and derivatives

Since this paper is concerned with partial differential equations which involve several independent and dependent variables, a simplified and completely rigorous treatment of the derivatives of smooth maps between vector spaces is a necessary prerequisite. This theory has been developed using the machinery of symmetric algebra, most notably in chapters one and three of H. Federer's book [9]. The advantages of this abstract machinery become clear in the crucial "Faa-di-Bruno formula" for the higher order partial derivatives of the composition of two maps. This section briefly reviews the relevant definitions, notations and results from symmetric algebra, most of which are discussed in much greater detail in [9]. A key new idea is the definition of the Faa-di-Bruno injection, which was motivated by the above mentioned formula, and is applied to provide an explicit matrix representation of the prolongations of the general linear group. To conclude this section, the notion of a total derivative is defined and discussed within the context of symmetric algebra; this will become important in later computations.

Let V be a real vector space. Let

$$\odot_* V = \bigoplus_{i=0}^{\infty} \odot_i V$$

denote the graded symmetric algebra based on V. The product in $\odot_* V$ will be denoted by the symbol $\circ$. Thus, if $v \in \odot_i V$ and $v' \in \odot_j V$, then their symmetric product $v \circ v' = v' \circ v \in \odot_{i+j} V$. If W is another real vector space, then let

$$\odot^*(V, W) = \bigoplus_{i=0}^{\infty} \odot^i(V, W)$$

denoted the graded vector space of W-valued symmetric linear forms on V. In other words, $\odot^i(V, W)$ is the vector space of all i-linear symmetric functions $A: \overbrace{V \times \cdots \times V}^{i} \to W$. There is a natural identification

$$\odot^i(V, W) \cong \mathrm{Hom}(\odot_i V, W).$$

Definition 2.1. Let

$$\mathbb{S}^n_k = \{I = (i_1, \cdots, i_n): 0 \leqslant i_\sigma \in \mathbf{Z}, \sigma = 1, \cdots, n;$$
$$\Sigma I = i_1 + \cdots + i_n = k\}$$

be the set of *n-multi-indices* of rank $k \geqslant 0$. Let

$$\mathbb{S}^n = \bigcup_{k=0}^{\infty} \mathbb{S}^n_k$$

be the set of all n-multi-indices. Given $I \in \mathbb{S}^n_k$, $J \in \mathbb{S}^n_l$ let $I + J \in \mathbb{S}^n_{k+l}$ be the multi-index with components $i_\sigma + j_\sigma$. Introduce a partial ordering on $\mathbb{S}^n$ by defining $I \leqslant J$ iff $i_\sigma \leqslant j_\sigma$ for all $\sigma = 1, \cdots, n$. In case $I \leqslant J$, let $J - I$ be the multi-index with components $j_\sigma - i_\sigma$. Define

$$I! = i_1! i_1! \cdots i_n!, \qquad I \in \mathbb{S}^n,$$
$$\binom{J}{I} = \frac{J!}{I!\,(J - I)!}, \qquad I \leqslant J \in \mathbb{S}^n.$$

Let $\delta^j \in \mathbb{S}^n_1$ be the multi-index with components δ^j_σ, the second δ being the Kronecker symbol.

Now suppose V is a finite dimensional real vector space with basis $\{e_1, \cdots, e_n\}$. In this case $\odot_k V$ has a corresponding basis given by

$$\{e_I: I \in \mathbb{S}_k^n\}, \text{ where } e_I = e_1^{i_1} \odot \cdots \odot e_n^{i_n}.$$

(The powers are of course taken in the symmetric algebra of V.) If W is a real algebra, then there is a naturally defined product on $\odot^*(V, W)$ making it into a graded commutative algebra. This product can be reconstructed from the following fundamental formula:

$$\phi \odot \psi(e_J) = \sum_{J > I \in \mathbb{S}_k^n} \binom{J}{I} \phi(e_I) \cdot \psi(e_{J-I}) \tag{2.1}$$

for $\phi \in \odot^k(V, W)$, $\psi \in \odot^l(V, W)$, $J \in \mathbb{S}_{k+l}^n$. More generally, it can be proven by induction from (2.1) that if $i_1, i_2, \cdots, i_m$ are nonnegative integers and $\phi_\sigma \in \odot^{i_\sigma}(V, W)$, $\sigma = 1, \cdots, m$, then for any $J \in \mathbb{S}_k^n$, $k = i_1 + \cdots + i_m$,

(2.1′)

$$\phi_1 \odot \phi_2 \odot \cdots \odot \phi_m(e_J) = \sum \frac{J!}{J_1! J_2! \cdots J_m!} \phi_1(e_{J_1}) \cdot \cdots \cdot \phi_m(e_{J_m}),$$

the sum being taken over all ordered sets of multi-indices $(J_1, J_2, \cdots, J_m)$ with $J_\sigma \in \mathbb{S}_{i_\sigma}^n$ and $J_1 + J_2 + \cdots + J_m = J$.

Now suppose that V and W are real normed vector spaces, and $f: V \to W$ is a smooth function. The k-th order differential of f at a point $x \in V$, which we will denote by the symbol $\partial^k f(x)$, is that symmetric k-linear W-valued form on V whose matrix entries are just the k-th order partial derivatives of f. (The reason for the symbol $\partial^k f$ rather than the more standard $d^k f$ or $D^k f$ will become clearer in what follows. Suffice it to say that $d^k f$ will be reserved for the action of a smooth map on the k-th order tangent bundle of a manifold, and $D^k f$ will be reserved for the total derivative.) In other words, we have $\partial^k f(x) \in \odot^k(V, W)$, and if $\{e_1, \cdots, e_n\}$ is a basis of V, then

$$\langle e_I, \partial^k f(x) \rangle = \partial_I f(x), \quad I \in \mathbb{S}_k^n.$$

Here $\partial_I = \partial_1^{i_1} \partial_2^{i_2} \cdots \partial_n^{i_n}$, ∂_i denoting the partial derivative in the e_i direction. Let

$$\odot^{k+l}(V, W) \subset \odot^l(V, \odot^k(V, W))$$

be the natural inclusion given by

$$\langle v, \langle v', w \rangle \rangle = \langle v \odot v', w \rangle, \; v \in \odot_l V, v' \in \odot_k V, w \in \odot^{k+l}(V, W).$$

It can be shown that under this inclusion,

$$\partial^l(\partial^k f)(x) = \partial^{k+l} f(x), x \in V.$$

Theorem 2.2 (*The Faa-di-Bruno formula*). *Let V, W, and X be normed real vector spaces, and let f: $V \to W$ and g: $W \to X$ be smooth maps. Given any $x \in V$, let $y = f(x) \in W$. For any positive integer k,*

$$\partial^k(g \circ f)(x) = \sum_{I \in \mathcal{Q}_k} \partial^{\Sigma I} g(y) \circ \left[\frac{1}{I!} \partial f(x)^{i_1} \odot \partial^2 f(x)^{i_2} \odot \cdots \odot \partial^k f(x)^{i_k} \right], \tag{2.2}$$

where

$$\mathcal{Q}_k = \left\{ I \in \mathbb{S}^k : \sum_{\sigma=1}^{k} \sigma i_\sigma = k \right\}.$$

Note that in formula (2.2) the powers and symmetric products of the differentials of f are taken in the algebra $\odot^*(V, \odot_* W)$. The proof of this theorem may be found in [9, p. 222].

For notational convenience, define

$$\odot_k^* V = \bigoplus_{i=1}^{k} \odot_i V,$$

$$\odot_*^k(V, W) = \bigoplus_{i=1}^{k} \odot^i(V, W).$$

There is a natural projection

$$\pi_k : \odot_*^k(V, \odot_* W) \to \odot_*^k(V, W)$$

given by composition with the projection $\odot_* W \to \odot_1 W = W$.

Definition 2.3. The *Faa-di-Bruno injection* is the map

$$\varepsilon_k : \odot_*^k(V, W) \to \odot_*^k(V, \odot_* W)$$

such that for a matrix $A \in \odot_*^k(V, W)$ with $A | \odot_i V = A_i$,

$$\varepsilon_k(A) = \sum_{j=1}^{k} \sum_{I \in \mathcal{Q}_j} \frac{1}{I!} A_1^{i_1} \odot A_2^{i_2} \odot \cdots \odot A_k^{i_k}. \tag{2.3}$$

Note that $\pi_k \circ \varepsilon_k$ is the identity map of $\odot_*^k(V, W)$.

Example 2.4. To get some idea of what the matrix $\varepsilon_k(A)$ looks like in block format, consider the case $k = 4$. Given $A \in \odot_*^4(V, W)$, then A has the block matrix form

$$A = (A_1 \ A_2 \ A_3 \ A_4).$$

Note that

$$\varepsilon_4(A) : \odot_4^* V \to \odot_4^* W,$$

by the definition of the set of multi-indices $\mathcal{Q}_4$. Using formula (2.3), we see that $\varepsilon_4(A)$ has block matrix form

$$\varepsilon_4(A) = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & \frac{1}{2}A_1\odot A_1 & A_1\odot A_2 & A_1\odot A_3 + \frac{1}{2}A_2\odot A_2 \\ 0 & 0 & \frac{1}{6}A_1\odot A_1\odot A_1 & \frac{1}{2}A_1\odot A_1\odot A_2 \\ 0 & 0 & 0 & \frac{1}{24}A_1\odot A_1\odot A_1\odot A_1 \end{bmatrix},$$

where the (i, j)th entry is the part of $\varepsilon_4(A)$ taking $\odot_j V$ to $\odot_i W$.

Returning to our discussion of the differentials of smooth functions between vector spaces, define

$$\partial_*^k f(x) = \partial f(x) + \partial^2 f(x) + \cdots + \partial^k f(x) \in \odot_*^k(V, W).$$

Then the Faa-di-Bruno formula (2.2) can be restated concisely as

$$\partial_*^k(g \circ f)(x) = \partial_*^k g(y) \circ \varepsilon_k\left[\partial_*^k f(x)\right]. \tag{2.4}$$

Define

$$d^k f(x) = \varepsilon_k\left[\partial_*^k f(x)\right].$$

(The reason for this notation will become clear in the context of higher order tangent vectors on manifolds; cf. Lemma 3.2.) Applying ε_k to (2.4) yields

$$d^k(g \circ f)(x) = d^k g(y) \circ d^k f(x). \tag{2.5}$$

Definition 2.5. Let V be a real normed vector space of dimension n, and let k be a positive integer. Let

$$\odot_*^k(V, V)_0 = \left\{A \in \odot_*^k(V, V) \colon A|V \in GL(V)\right\}$$

be the set of symmetric linear functions from V to itself of order $\leqslant k$ which are invertible when restricted to $\odot_1 V = V$. Define the *k-th prolongation of the general linear group of* V to be

$$GL^{(k)}(V) = \varepsilon_k\left[\odot_*^k(V, V)_0\right],$$

which is a matrix subgroup of $GL(\odot_k^* V)$.

Note that this definition coincides with that given in [13, p. 139], since $GL^{(k)}(V)$ can be realized as the set of all matrices $d^k f(0)$ corresponding to all local diffeomorphisms f of V with $f(0) = 0$, the group multiplication being induced by composition of diffeomorphisms according to (2.5). In addition, we have given an explicit matrix representation of $GL^{(k)}(V)$. In fact, given $A \in GL^{(k)}(V)$, let (A_j^i) be the block matrix form of A so that $A_j^i \colon \odot_j V \to \odot_i V$. By the definition, A has the following properties:

(i) A is block upper triangular; i.e., $A_j^i = 0$ for $i > j$.

(ii) Let

$$\mathcal{Q}_j^i = \{I \in \mathcal{Q}_j : \Sigma I = i\} = \{I \in \mathbb{S}^j : \Sigma I = i, \sum \sigma i_\sigma = j\}.$$

Then

$$A_j^i = \sum_{I \in \mathcal{Q}_j^i} \frac{1}{I!} (A_1^1)^{i_1} \odot (A_2^1)^{i_2} \odot \cdots \odot (A_k^1)^{i_k}. \tag{2.6}$$

(iii) If $A \in GL^{(k)}(V)$, then for $l < k$, $A | \odot_l^* V \in GL^{(l)}(V)$.

In particular, if $A = d^k f(x)$, then we will abbreviate $A_j^i = \partial_j^i f(x)$. This gives $\partial_j^1 f(x) = \partial^j f(x)$, which is a little confusing, but the symbol ∂_j has been reserved for the partial derivative in the x^j direction.

Definition 2.6. Let X be a fixed real normed vector space. Let U and W be normed vector spaces and let $Z = X \times U$. Suppose $F: Z \to W$ is a smooth function. The *k-th order total differential* of F is the unique map

$$D^k F: Z \times \odot_*^k(X, U) \to \odot^k(X, W)$$

such that for any smooth $f: X \to U$,

$$D^k F\big(x, f(x), \partial_*^k f(x)\big) = \partial^k[F \circ (\mathbf{1}_X \times f)](x), \tag{2.7}$$

where $\mathbf{1}_X$ denotes the identity map of X.

Existence and uniqueness of the total differential follows from the Faa-di-Bruno formula. In fact,

$$D^k F\big(x, f(x), \partial_*^k f(x)\big) = \sum_{l=1}^{k} \partial^k F(x, f(x)) \cdot \partial_l^k(\mathbf{1} \times f)(x),$$

where $\partial_l^k(\mathbf{1} \times f)$ are the matrix blocks in $d^k(\mathbf{1} \times f)$ and are thus expressed in terms of $\partial_*^k f(x)$. Let

$$D_*^k F = D^1 F + D^2 F + \cdots + D^k F: Zx\odot_*^k(X, U) \to \odot_*^k(X, W).$$

In the case that X and U are finite dimensional, with respective bases $\{e_1, \cdots, e_p\}$ and $\{e'_1, \cdots, e'_q\}$, the matrix elements of $D^k F(z, u^{(k)})$ where $z \in Z$, $u^{(k)} \in \odot_*^k(X, U)$, are given by

$$D_I F(z, u^{(k)}) = \langle e_I, D^k F(z, u^{(k)}) \rangle, \quad I \in \mathbb{S}_k^n.$$

Given an element $u^{(k)} \in \odot_*^k(X, U)$ we will let $u^{(l)} \in \odot_*^l(X, U)$ denote its restriction to $\odot_l^* X$. The coordinates of $u^{(k)}$ are given by

$$\sum_{l=1}^{q} u_J^l e'_l = \langle e_J, u^{(k)} \rangle \text{ for } k \geqslant \sum J.$$

Lemma 2.7. *Let $F\colon X \times U \to W$ be smooth. Then the matrix entries of $D^{k+1}F$ are given recursively by*

$$D_{I+\mathcal{E}^j}F(x, u, u^{(k+1)}) = D_j\left[D_I F(x, u, u^{(k)})\right] \tag{2.8}$$

for $I \in \mathbb{S}_k$, $1 \leqslant j \leqslant p$, $x \in X$, $u \in U$, $u^{(k+1)} \in \odot_*^{k+1}(X, U)$. Here D_j is the total derivative operator in the e_j direction, given by

$$D_j = \partial_j + \sum_{l=1}^{q}\left\{u_j^l\frac{\partial}{\partial u^l} + \sum_{0<I\in\mathbb{S}^n} u_{I+\mathcal{E}^j}^l\frac{\partial}{\partial u_I^l}\right\}. \tag{2.9}$$

(Note that for any fixed k only finitely many terms in the expression for D_j are necessary.)

This lemma provides the means for actually computing the total derivative. The proof is a straightforward induction argument; cf. [16, p. 98].

3. Extended jet bundles

The first step in the development of a comprehensive theory of systems of partial differential equations over arbitrary smooth manifolds representing both the independent and dependent variables is the construction of an appropriate fiber bundle over the manifold whose points represent the partial derivatives of the dependent variables of order $\leqslant k$, called the extended k jet bundle of the manifold. To give some motivation for the definitions to follow, the construction of the jet bundles of a vector bundle will be briefly recalled following [10]. Then some preliminary definitions for the more general case will be made; the machinery needed to fully explore these ideas will be developed in subsequent sections.

Suppose $\pi\colon Z \to X$ is a vector bundle over a p-dimensional base manifold X, representing the independent variables, with q-dimensional fiber U, representing the dependent variables. Sections of Z are usually defined to be smooth maps $f\colon X \to Z$ such that $\pi \circ f = 1_X$, the identity map of X. It will be more convenient, however, to view sections geometrically as p-dimensional submanifolds of Z which satisfy a condition of transversality to the fibers of Z. In addition, for the submanifold to truly be a section, it must satisfy a further global condition of intersecting each fiber exactly once; in the construction of the jet bundles, this condition can be safely ignored, since only local sections are needed. The k-jet bundle J_kZ is given by the equivalence classes of sections of Z having k-th order contact.

Definition 3.1. Let M be a smooth manifold and let $m \in M$. Let $C^\infty(M, \mathbf{R})|_m$ denote the algebra of germs of smooth real valued functions on M at the point m. Let $I_m \subset C^\infty(M, \mathbf{R})|_m$ be the ideal of germs of functions which vanish at m, and let I_m^k denote its k-th power, which consists of all finite linear combinations of k-fold products of elements of I_m. The *k-th order cotangent bundle* of M at m is

$$\mathfrak{T}_k^* M|_m = I_m / I_m^{k+1}.$$

The *k-th order tangent bundle* to M at m is the dual space

$$\mathfrak{T}_k M|_m = [\mathfrak{T}_k^* M|_m]^*.$$

In local coordinates, the fibers of the bundle $\mathfrak{T}_k M$ are

$$\mathfrak{T}_k M|_m \simeq \odot_k^* TM|_m = \bigoplus_{i=1}^{k} \odot_i TM|_m.$$

(The details of this construction are given in [24, §1.26].) If $\{\partial_1, \cdots, \partial_n\}$ is a basis of $TM|_m$, then

$$\{\partial_I\colon\ I = (i_1, \cdots, i_n), \Sigma I = i_1 + \cdots + i_n \leqslant k\}$$

forms a basis of $\mathfrak{T}_k M|_m$.

Submanifolds S and S' of M have k-th order contact at $m \in S \cap S'$ if $\mathfrak{T}_k S|_m = \mathfrak{T}_k S'|_m$. It can be readily checked that in the case of sections of a vector bundle this definition of k-th order contact agrees with other definitions, and has the advantage of an immediate geometrical interpretation.

Suppose M and M' are smooth manifolds, and $F\colon M \to M'$ is a smooth map. There is an induced bundle morphism $d^kF\colon \mathfrak{T}_k M \to \mathfrak{T}_k M'$ given by the formula $d^kF(\mathbf{v})(f) = \mathbf{v}(f \circ F)$ for $\mathbf{v} \in \mathfrak{T}_k M$ and $f \in C^\infty(M', \mathbf{R})$. It is readily verified that if $G\colon M' \to M''$ is another smooth map, then

$$d^kG \circ d^kF = d^k(G \circ F). \tag{3.1}$$

Lemma 3.2. *Let $F\colon V \to W$ be a smooth map between finite dimensional real vector spaces. Let $x \in V$ and $w = F(x)$. Then the map*

$$d^kF|_x\colon \mathfrak{T}_k V|_x \to \mathfrak{T}_k W|_w,$$

under the identification $\mathfrak{T}_k V|_x \simeq \odot_k^* V$, is the same as the map

$$d^kF(x) = \varepsilon_k\Big[\partial_*^k F(x)\Big]\colon \odot_k^* V \to \odot_k^* W.$$

Proof. Let $\{e_1, \cdots, e_n\}$ and $\{e_1', \cdots, e_m'\}$ be bases of V and W respectively. The identification $\mathfrak{T}_k V|_x \simeq \odot_*^k V$ takes $\partial_I|_x$ to e_I for any multi-index I. Now suppose $f \in C^\infty(W, \mathbf{R})$. Then

$$d^kF(\partial_I|_x)f = \partial_I(f \circ F)(x) = \sum_{\Sigma I \leqslant \Sigma J} \partial_J' f(w) \cdot \Big[\varepsilon_k \partial_*^k F(x)\Big]_I^J,$$

which follows from the Faa-di-Bruno identity. (For a matrix $A\colon \odot_k^* V \to \odot_k^* W$, A_I^J is the matrix entry given by $A(e_I) = \sum A_I^J e_J'$.) Therefore

$$d^k F(\partial_I|_x) = \sum \left[\varepsilon_k \partial_*^k F(x) \right]_I^J \cdot \partial_J'|_w.$$

Notice that $\partial_*^k F(x)$ has matrix entries $\partial_I F^j(x)$ for $j = 1, \cdots, m$ and $1 \leqslant \Sigma I \leqslant k$ showing that $d^k F$ is uniquely determined by the partial derivatives of F of order $\leqslant k$. If F: $Z \to Z'$ is a smooth map between smooth manifolds then the local coordinate descriptions of Z and Z' identify them with open subsets of real vector spaces, so the local coordinate description of $d^k F$ is given by the Faa-di-Bruno formula.

Definition 3.3. Given a smooth manifold Z and a point $z \in Z$, a *p-section* of Z passing through z is an arbitrary smooth p-dimensional submanifold of Z containing z. The space of *germs* of p-sections of Z passing through z, $C^\infty(Z, p)|_z$, is the set of all smooth p-dimensional submanifolds of Z passing through z modulo the equivalence relation that s and s' define the same germ at z iff there is a neighborhood V of z with $s \cap V = s' \cap V$.

Definition 3.4. The *space of extended k-jets of p-sections* of Z at a point $z \in Z$, $J_k^*(Z, p)|_z$, is given by the space of germs of p-sections of Z passing through z modulo the equivalence relation of k-th order contact.

A p-section s of Z can be described locally by a smooth imbedding f: $U \to Z$, where U is an open subset of $\mathbf{R}^p$, called a local *parametrization* of s. In general, since all considerations here are of a local nature, we will be a bit sloppy notationally and write f: $\mathbf{R}^p \to Z$ even though f might only be defined on an open subset of $\mathbf{R}^p$. Note that two embeddings f and g are parametrizations of the same germ at z iff there exists a (local) diffeomorphism ψ: $\mathbf{R}^p \to \mathbf{R}^p$ such that $f \circ \psi = g$ near the point z. The notation $j_k^* f|_z$ will mean the extended k-jet of the p-section given by im f at z.

Define the extended k-jet bundle of p-sections of Z to be

$$J_k^*(Z, p) = \bigcup_{z \in Z} J_k^*(Z, p)|_z.$$

For any smooth p-dimensional submanifold $s \subset Z$, define its extended k-jet to be

$$j_k^* s = \bigcup_{z \in s} j_k^* s|_z,$$

which will subsequently be shown to be a smooth p-dimensional submanifold of $J_k^*(Z, p)$. The next step is to describe the fiber bundle structure and local coordinate description of $J_k^*(Z, p)$; its fibers are "prolonged Grassmann manifolds", which we proceed to define and characterize.

Let V be a real normed vector space and let 0 denote the origin in V. The identifications $TV|_0 = V$, $\mathfrak{T}_k V|_0 \simeq \odot_k^* V$ will be used without further comment.

Definition 3.5. The k-th order *prolonged Grassmann manifold* of *prolonged p-planes* in V is $\mathrm{Grass}^{(k)}(V,p) = \{\mathfrak{T}_k s|_0 \subset \odot_k^* V : s$ is a smooth p dimensional submanifold of V passing through $0\}$.

Lemma 3.6. *Let $\odot_*^k(\mathbf{R}^p, V)_0$ denote the open subset of $\odot_*^k(\mathbf{R}^p, V)$ consisting of those maps $A\colon \odot_k^*\mathbf{R}^p \to V$ whose restriction to $\odot_1\mathbf{R}^p$ has maximal rank equal to p. There is a natural action of $GL^{(k)}(p)$ on $\odot_*^k(\mathbf{R}^p, V)_0$ given by right multiplication of matrices, and*

$$\mathrm{Grass}^{(k)}(V,p) \simeq \odot_*^k(\mathbf{R}^p, V)_0 / GL^{(k)}(p).$$

Proof. Given a p dimensional submanifold $s \subset V$ passing through 0, let $f\colon \mathbf{R}^p \to V$ be a local parametrization of s near 0 with $f(0) = 0$. Since f is an embedding,

$$d^k f[\mathfrak{T}_k \mathbf{R}^p|_0] = \mathfrak{T}_k s|_0, \quad \partial_*^k f(0) \in \odot_*^k(\mathbf{R}^p, V)_0.$$

Therefore it suffices to show that if f and f' are smooth embeddings of $\mathbf{R}^p$ into V with $f(0) = f'(0) = 0$, then

$$(*) \qquad \mathrm{im}\, d^k f|_0 = \mathrm{im}\, d^k f'|_0$$

iff there exists a matrix $A \in GL^{(k)}(p)$ satisfying

$$\partial_*^k f(0) \cdot A = \partial_*^k f'(0).$$

First, given such a matrix A, $(*)$ follows immediately since A is invertible on $\odot_k^*\mathbf{R}^p = \mathfrak{T}_k\mathbf{R}^p|_0$. Conversely, given $(*)$, let $V' \subset V$ be an $(n-p)$-dimensional subspace such that in a neighborhood of the origin $T(\mathrm{im}\, f) \cap V' = T(\mathrm{im}\, f') \cap V' = \{0\}$. Let $\pi\colon V \to V/V' = \mathbf{R}^p$ be the projection, so by the inverse function theorem both $\pi \circ f$ and $\pi \circ f'$ are invertible in a neighborhood of the origin in $\mathbf{R}^p$. Now since $(*)$ holds, there exists a linear transformation $A\colon \odot_k^*\mathbf{R}^p \to \odot_k^*\mathbf{R}^p$ satisfying

$$d^k f|_0 \cdot A = d^k f'|_0.$$

Therefore

$$d^k\pi \cdot d^k f \cdot A = d^k \pi \cdot d^k f',$$
$$A = d^k(\pi \circ f')|_0 \cdot d^k(\pi \circ f)^{-1}|_0,$$

proving that $A \in GL^{(k)}(p)$.

For positive integers $k > l$ there is a canonical projection

$$\pi_l^k\colon \mathrm{Grass}^{(k)}(V,p) \to \mathrm{Grass}^{(l)}(V,p),$$

given by

$$\pi_l^k[\mathfrak{T}_k s|_0] = \mathfrak{T}_l s|_0.$$

It will be shown that π_l^k makes $\mathrm{Grass}^{(k)}(V, p)$ into a Euclidean fiber bundle over $\mathrm{Grass}^{(l)}(V, p)$. Suppose $\Lambda \in \mathrm{Grass}^{(k)}(V, p)$ and $\{e_1, \cdots, e_p\}$ is a basis for the p-plane $\Lambda_1 = \pi_1^k(\Lambda) \in \mathrm{Grass}(V, p)$. (Given Λ as an abstract subspace of $\odot_k^* V$, Λ_1 is the unique p-plane in V such that $\odot_k \Lambda_1 = \Lambda \cap \odot_k V$.) Let π_Λ: $V \to V/\Lambda_1$ be the projection. It is claimed that there is a unique set of elements

$$\{e_J' \in V/\Lambda_1 : J \in \mathcal{S}^p, 1 < \Sigma J \leqslant k\}$$

which have the property that for any $e_J \in V$ with $\pi_\Lambda(e_J) = e_J'$, the vectors

$$\hat{e}_J = \sum \frac{J!}{J_1! J_2! \cdots J_l!} e_{J_1} \odot \cdots \odot e_{J_l} \tag{3.2}$$

for $1 \leqslant \Sigma J \leqslant l$ form a basis for Λ. The summation in (3.2) is taken over all unordered sets of nonzero multi-indices $\{J_1, \cdots, J_l\}$ with $J_1 + J_2 + \cdots + J_l = J$. The proof of this claim is a direct consequence of the Faa-di-Bruno formula. Namely, if f: $\mathbf{R}^p \to V$ is an embedding such that

$$\mathfrak{T}_k(\mathrm{im}\, f)|_0 = d^k f(\mathfrak{T}_k \mathbf{R}^p|_0) = \Lambda,$$

then let

$$e_J = \partial^{\Sigma J} f(\partial_J|_0).$$

We conclude that

$$\hat{e}_J = d^k f(\partial_J|_0)$$

by applying formula (2.1′) to the Faa-di-Bruno formula.

Conversely, given a basis $\{e_1, \cdots, e_p\}$ of a p-plane $\Lambda_1 \subset V$ and elements $e_J' \in V/\Lambda_1$ for $J \in \mathcal{S}^p$ with $1 < \Sigma J \leqslant k$, it is not hard to see that the subspace spanned by $\{\hat{e}_J : 1 \leqslant \Sigma J \leqslant k\}$ as given by (3.2) is an element of $\mathrm{Grass}^{(k)}(V, p)$. In fact, if $\{\varepsilon_1, \cdots, \varepsilon_n\}$ is a basis for V and $e_J = \sum c_J^i \varepsilon_i$, then the polynomial f: $\mathbf{R}^p \to V$,

$$f^i(x^1, \cdots, x^p) = \sum c_J^i \frac{x^J}{J!}$$

satisfies $\mathfrak{T}_k(\mathrm{im}\, f)|_0 = \Lambda$. Moreover Λ is uniquely determined by the e_J' once $e_1, \cdots, e_p$ are prescribed. Let

$$N_{k,p} = \dim \odot_k^* \mathbf{R}^p = \binom{p+k}{k} - 1. \tag{3.3}$$

We have thus proven the following.

Proposition 3.7. *Let Q denote the standard quotient bundle over* $\mathrm{Grass}(V, p)$. *Then*

$$\mathrm{Grass}^{(k)}(V, p) \simeq (N_{k,p} - p)Q \tag{3.4}$$

are diffeomorphic as smooth manifolds.

Remark. If U denotes the universal bundle over Grass(V, p), whose fiber over a p-plane Λ is just Λ itself, and I denotes the trivial bundle $V \times$ Grass(V, p), then Q is given as the quotient bundle

$$0 \to U \to I \to Q \to 0,$$

whose fiber over a p-plane Λ is the quotient space V/Λ. The notation jQ for j an integer just means $Q \oplus \cdots \oplus Q$ j times.

In particular, this proposition shows that the bundle

$$\pi_l^k\colon \mathrm{Grass}^{(k)}(V, p) \to \mathrm{Grass}^{(l)}(V, p)$$

has Euclidean fiber of dimension $N_{k,p} - N_{l,p}$. These bundles are not necessarily trivial. For instance

$$\mathrm{Grass}^{(2)}(2, 1) \to \mathrm{Grass}(2, 1) \simeq S^1$$

is the Möbius line bundle over the circle S^1.

For computational purposes, some natural coordinate charts on Grass$^{(k)}(n, p)$ similar to the standard coordinate charts on Grass(n, p) will be introduced. Given a matrix $A \in \odot_*^k(\mathbf{R}^p, \mathbf{R}^n)_0$ let $A^j = A|\odot_j\mathbf{R}^p$ so that A has the block matrix form

$$A = (A^1\, A^2 \cdots A^k),$$

where each A^j is an $n \times \binom{p+j-1}{j}$ matrix. Let $\{A_\alpha^1\}$ denote the set of all minors of the matrix A^1, where for $\alpha = (\alpha_1, \cdots, \alpha_p)$ the minor A_α^1 is the $p \times p$ matrix consisting of rows $\alpha_1, \cdots, \alpha_p$ of A^1. Similarly, let A_α and A_α^j denote the matrices consisting of rows $\alpha_1, \cdots, \alpha_p$ of A and A^j. Let

$$\Pi\colon \odot_*^k(\mathbf{R}^p, \mathbf{R}^n)_0 \to \mathrm{Grass}^{(k)}(n, p)$$

be the projection as given in Lemma 3.6. Let

$$U_\alpha = \Pi\{A\colon \det A_\alpha^1 \neq 0\},$$

which is an open subset of Grass$^{(k)}(n, p)$. The U_α's cover Grass$^{(k)}(n, p)$.

Given $A \in \odot_*^k(\mathbf{R}^p, \mathbf{R}^p)_0$ with A_α^1 nonsingular, there is a unique matrix $K \in GL^{(k)}(p)$ such that the matrix $B = AK$ is of the form

$$B_\alpha = (\mathbf{1}_p\, 0\, 0 \cdots 0).$$

In fact, K is found by recursion as follows. Suppose

$$K = \varepsilon_k(K^1\, K^2 \cdots K^k).$$

Then the Faa-di-Bruno formula shows that

$$(3.5)\qquad \begin{aligned} K^1 &= \left(A^1_\alpha\right)^{-1},\\ K^2 &= -K^1\cdot A^2_\alpha\cdot\frac{1}{2}K^1\odot K^1,\\ K^3 &= -K^1\cdot\left[A^2_\alpha\cdot K^1\odot K^2 + A^3_\alpha\cdot\frac{1}{3!}K^1\odot K^1\odot K^1\right],\\ &\ \vdots\qquad\qquad\vdots\\ K^k &= -K^1\cdot\sum_{j=2}^{k}A^j_\alpha\sum_{I\in\mathscr{Q}^j_k}\frac{1}{I!}(K^1)^{i_1}\odot\cdots\odot(K^k)^{i_k}\end{aligned}$$

(Note that no term on the right hand side of the last equation actually contains K^k.) This procedure gives a well-defined map

$$h_\alpha\colon U_\alpha\xrightarrow{\simeq}\mathbf{R}^{N_{k,p}}$$

where $h_\alpha[\Pi A]$ is the matrix $\hat{B}_\alpha$ consisting of the rows of B not in B_α. Moreover, it is easy to see using (3.5) that the transition functions $h_\beta\circ h_\alpha^{-1}$ are smooth maps, so the U_α's do indeed form a coordinate atlas on $\mathrm{Grass}^{(k)}(n,p)$.

The preceding construction of local coordinates is just a special case of the trivialization of the prolonged Grassman manifolds.

Lemma 3.8. *Let $W\subset V$ be an $(n-p)$-dimensional subspace. Then the trivialized prolonged Grassmannian with respect to W is*

$$\mathrm{Grass}^{(k)}(V,p;\,W)=\{\mathcal{T}_k s|_0\colon Ts|_0\cap W=\{0\}\},$$

and is diffeomorphic to $\mathbf{R}^{N_{k,p}}$.

The trivialized Grassmannian $\mathrm{Grass}^{(k)}(V,p;\,W)$ is just the space of k-th order tangent spaces of sections transversal to W. For the above coordnate charts, $U_\alpha=\mathrm{Grass}^{(k)}(V,p;\,W_\alpha)$ where W_α is the orthogonal complement to the subspace spanned by $\{e_{\alpha_1},\cdots,e_{\alpha_p}\}$.

There is a natural action of the Lie group $GL^{(k)}(n)$ induced by the action of diffeomorphisms of $\mathbf{R}^n$ on sections. Namely, if $A\in GL^{(k)}(n)$ and $\Lambda\in\mathrm{Grass}^{(k)}(n,p)$, then given any p-section $s\in C^\infty(\mathbf{R}^n,p)|_0$ with $\mathcal{T}_k s|_0=\Lambda$ and a local diffeomorphism $G\colon\mathbf{R}^n\to\mathbf{R}^n$ with $G(0)=0$, $d^kG(0)=A$, then

$$A\cdot\Lambda=\mathcal{T}_k[G(s)]|_0.$$

This just corresponds to left matrix multiplication

$$A\cdot\Lambda=\Pi(A\cdot B)=\Pi(\pi_k(A\cdot\varepsilon_k B)),\quad B\in\odot^k_*(\mathbf{R}^p,\mathbf{R}^n)_0,$$

where π_k is the projection inverse to the Faa-di-Bruno injection ε_k and B is any matrix such that $\Pi\,B=\Lambda$. (Note that the action of $GL^{(k)}(n)$ commutes with the action of $GL^{(k)}(p)$ on $\odot^k_*(\mathbf{R}^p,\mathbf{R}^n)$.) $GL^{(k)}(n)$ acts transitively on $\mathrm{Grass}^{(k)}(n,p)$.

Now suppose that Z is a smooth manifold, and $E \to Z$ is a bundle with fiber $\odot_k^* \mathbf{R}^n$ and group $GL^{(k)}(n)$. (For example, $E = \mathfrak{T}_k Z$.) By the general theory of fiber bundles, for $0 < p < n$ there exists a unique bundle $\mathrm{Grass}^{(k)}(E, p) \to Z$ having fiber $\mathrm{Grass}^{(k)}(E|_z, p)$ over $z \in Z$ such that if $E_0 \subset E$ is any smooth (local) subbundle of prolonged p-planes, then E_0 defines a smooth (local) section of $\mathrm{Grass}^{(k)}(E, p)$. If the transition functions of E are given by $A_{\alpha\beta} \in GL^{(k)}(n)$, then the transition functions for $\mathrm{Grass}^{(k)}(E, p)$ are given by the images of the $A_{\alpha\beta}$ in $PGL^{(k)}(n)$, the k-th prolongation of the projective linear group, which is obtained as the quotient group of $GL^{(k)}(n)$ by its center–the group consisting of all nonzero multiplies of the identity map of $\odot_k^* \mathbf{R}^n$. Actually, to do the preceding construction, we need the following lemma.

Lemma 3.9. *The action of $PGL^{(k)}(n)$ on* $\mathrm{Grass}^{(k)}(n, p)$ *for $0 < p < n$ is effective; i.e., if $A \in GL^{(k)}(n)$ is such that $A \cdot \Lambda = \Lambda$ for all prolonged p-planes $\Lambda \in \mathrm{Grass}^{(k)}(n, p)$, then $A = \lambda \mathbf{1}$ for some $\lambda \in \mathbf{R}$.*

The proof is a direct consequence of the corresponding statement for ordinary Grassmannians and the following elementary lemma from symmetric algebra.

Lemma 3.10. *If V is a real finite dimensional vector space, then*

$$\{ v \odot v \odot \cdots \odot v \in \odot_k V : v \in V \}$$

spans $\odot_k V$.

As a corollary of these more abstract considerations, we obtain an alternate characterization of the extended jet bundles of a smooth manifold as appropriate prolonged Grassmann bundles. This is perhaps the most convenient characterization of the bundle structure of the extended jet bundles.

Proposition 3.11. *There is an identification*

$$J_k^*(Z, p) \simeq \mathrm{Grass}^{(k)}(\mathfrak{T}_k Z, p)$$

giving $J_k^*(Z, p)$ the structure of a fiber bundle over Z with fiber $\mathrm{Grass}^{(k)}(n, p)$ *where $n =$* dim *Z, and group $PGL^{(k)}(n)$, such that if $s \subset Z$ is any smooth p-dimensional submanifold, then $j_k^* s \subset J_k^*(Z, p)$ is also a smooth p-dimensional submanifold.*

Proposition 3.12. *Let $k > l$ be positive integers. Then*

$$\pi_l^k : J_k^*(Z, p) \to J_l^*(Z, p)$$

is a fiber bundle with Euclidean fiber of dimension $N_{k,p} - N_{l,p}$.

Suppose that $\mathfrak{U}$ is an involutive $(n-p)$-dimensional differential system on the n-dimensional manifold Z. The trivialized extended k-jet bundle of Z with respect to $\mathfrak{U}$ is the open subbundle

$$J_k^*(Z, p; \mathfrak{U}) = \mathrm{Grass}^{(k)}(\mathfrak{T}_k Z, p; \mathfrak{U})$$

consisting of the k-th order tangent spaces of sections transverse to $\mathfrak{U}$. By Lemma 3.8, $J_k^*(Z, p; \mathfrak{U})$ is a bundle with Euclidean fiber of dimension $N_{k,p}$. We now propose to introduce "canonical" coordinates on $J_k^*(Z, p; \mathfrak{U})$ associated with a coordinate system on Z which is flat with respect to $\mathfrak{U}$.

Let χ: $Z \to \mathbf{R}^p \times \mathbf{R}^q$ be a local coordinate system on Z with the coordinates on $\mathbf{R}^p \times \mathbf{R}^q$ denoted by $(x, u) = (x^1, \cdots, x^p, u^1, \cdots, u^q)$ so that the differential system $\mathfrak{U}$ is spanned by $\{\partial/\partial u^1, \cdots, \partial/\partial u^q\}$. The case to keep in mind is when $Z \to X$ is a vector bundle and $\mathfrak{U}$ is the differential system given by the tangent spaces to the fibers, so that $(x^1, \cdots, x^p)$ are the coordinates on the base manifold X (independent variables) and $(u^1, \cdots, u^q)$ are the fiber coordinates (dependent variables). In this case we can identify the trivialized jet bundle with the ordinary jet bundle corresponding to the vector bundle Z, since both are constructed by consideration of p-sections transverse to the fibers.

Proposition 3.13. *Let $Z \to X$ be a fiber bundle over a p-dimensional manifold X, and let $\mathfrak{U}$ denote the involutive differential system of tangent spaces to the fibers of Z. Then*

$$J_k^*(Z, p; \mathfrak{U}) \simeq J_k Z.$$

This shows that the extended jet bundle can be regarded as the "completion" of the ordinary jet bundle in the same manner that projective space is the completion of affine space. In this case the completion is obtained by allowing sections with vertical tangents.

Let s be a p-section of Z transverse to the differential system $\mathfrak{U}$. The *normal parametrization* of s relative to the coordinate system χ is that map $\hat{f}$: $\mathbf{R}^p \to Z$ with $\operatorname{im} \hat{f} = s$ and $\chi \circ \hat{f} = \mathbf{1}_p \times f$ for some (uniquely determined) smooth f: $\mathbf{R}^p \to \mathbf{R}^q$. Then

$$\partial_*^k(\chi \circ f)(x) = \begin{pmatrix} \mathbf{1}_p & 0 & \cdots & 0 \\ \partial f(x) & \partial^2 f(x) & \cdots & \partial^k f(x) \end{pmatrix},$$

so that $\partial_*^k f(x)$ can be regarded as the local coordinates of the extended k-jet of s at the point $f(x)$.

Proposition 3.14. *Let χ: $Z \to \mathbf{R}^p \times \mathbf{R}^q$ be a local coordinate system on the smooth manifold Z, and let $\mathfrak{U}$ denote the local differential system $(d\chi)^{-1}T\mathbf{R}^q$. Then there is an induced local coordinate system*

$$\chi^{(k)}: J_k^*(Z, p; \mathfrak{U}) \to \mathbf{R}^p \times \mathbf{R}^q \times \odot_*^k(\mathbf{R}^p, \mathbf{R}^q),$$

such that if $\hat{f}$: $\mathbf{R}^p \to Z$ is the normal parametrization of a p-section of Z with $\chi \circ \hat{f} = \mathbf{1}_p \times f$, then

$$\chi^{(k)} \circ j_k^* \hat{f} = \mathbf{1}_p \times f \times \partial_*^k f.$$

If (x, u) are the local coordinates on Z, then the local coordinates on $J_k^*(Z, p)$ will usually be denoted by $(x, u, u^{(k)})$. Here $u^{(k)}$ is a matrix with entries u_K^i for $1 \leqslant i \leqslant q$ and $K \in \mathbb{S}^p$, $1 \leqslant \Sigma K \leqslant k$, so that if $u^i = f^i(x)$, then $u_K^i = \partial_K f^i(x)$. Note that if $\hat{g}$: $\mathbf{R}^p \to Z$ is any parametrization of a p-section of Z transversal to $\mathcal{U}$ so that $\chi \circ \hat{g} = g_1 \times g_2$, then g_1 is locally invertible, so the normal parametrization of im $\hat{g}$ is given by $\chi^{-1} \circ (\mathbf{1}_p \times g_2 \circ g_1^{-1})$, and

$$\chi^{(k)} \circ j_k^* g = \mathbf{1}_p \times g_2 \circ g_1^{-1} \times \partial_*^k (g_2 \circ g_1^{-1}).$$

4. Differential operators and equations

The next step in our theory of extended jet bundles over smooth manifolds is to describe what is meant by a differential operator and a differential equation in this context. These concepts should be direct generalizations of the corresponding objects for vector bundles and should include as special cases what are classically meant by systems of partial differential equations. First, a differential operator on an extended jet bundle will be defined and some important properties described. Next, we proceed to a discussion of differential equations, which will be connected with the previously mentioned differential operators. Recall that a differential operator in the category of vector bundles is given by a smooth fiber-preserving map from a jet bundle to another vector bundle. In strict analogy we make the definition of a differential operator on an arbitrary smooth manifold.

Definition 4.1. Let Z be a smooth manifold of dimension $p + q$. A *k-th order differential operator* (for p-sections) is a fiber bundle morphism

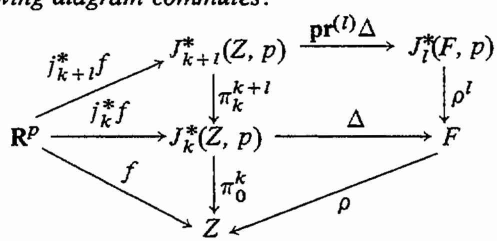
$$\Delta \colon J_k^*(Z, p) \to F,$$

where $\rho \colon F \to Z$ is a fiber bundle over Z, such that $\rho \circ \Delta$ is the projection π_0^k: $J_k^*(Z, p) \to Z$.

Proposition 4.2. *Let* $\Delta \colon J_k^*(Z, p) \to F$ *be a k-th order differential operator, and let l be a nonnegative integer. Then there exists* $(k + l)$*-th order differential operator*

$$\mathrm{pr}^{(l)}\Delta \colon J_{k+l}^*(Z, p) \to J_l^*(F, p)$$

called the l-th prolongation of Δ*, such that for any p-section parametrized by f:* $\mathbf{R}^p \to Z$ *the following diagram commutes:*



In terms of local coordinates (x, u) on Z and (x, u, w) on F,

$$\mathbf{pr}^{(l)}\Delta(x, u, u^{(k+l)}) = \left(x, u, \Delta(x, u, u^{(k)}), D_*^l\Delta(x, u, u^{(k+l)})\right)$$

for $(x, u) \in Z$, $(x, u, u^{(k+l)}) \in J_{k+l}^*(Z, p)$ *with* $(x, u, u^{(k)}) = \pi_k^{k+l}(x, u, u^{(k+l)})$.

The proof is a straightforward generalization of the proof of the corresponding proposition for ordinary jet bundles.

Corollary 4.3. *Let*

$$\Delta = \mathbf{1}: J_k^*(Z, p) \to J_k^*(Z, p)$$

be the identity map. Then for nonnegative integers l there is a natural embedding

$$\iota_k^{k+l} = \mathbf{pr}^{(l)}\mathbf{1}: J_{k+l}^*(Z, p) \hookrightarrow J_l^*(J_k^*(Z, p), p).$$

In terms of local coordinates (x, u) on Z and the induced coordinates $(x, u, u^{(k+l)})$ on $J_{k+l}^*(Z, p)$ and $(x, u, u^{(k)}, u^{(l)}, (u^{(k)})^{(l)})$ on $J_l^*(J_k^*(Z, p), p)$, $J_{k+l}^*(Z, p)$ is the subbundle of $J_l^*(J_k^*(Z, p), p)$ given by

$$\big\{(u_I^i)_J = (u_{I'}^i)_{J'}: I + J = I' + J',\ I, J, I', J' \in \mathcal{S}^p,$$
$$0 \leqslant \Sigma I, \Sigma I' \leqslant k, 0 \leqslant \Sigma J, \Sigma J' \leqslant l, i = 1, \cdots, q\big\},$$

where u_0^i denotes the coordinate u^i, and $(u_I^i)_0$ the coordinate u_I^i.

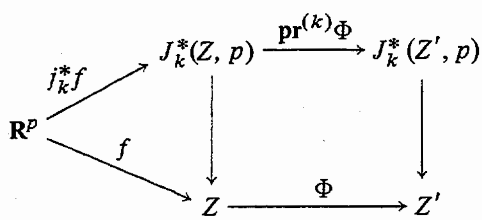
Corollary 4.4. *If Δ, k, l are as in the proposition, and l' is another nonnegative integer, then*

$$\mathbf{pr}^{(l')}\mathbf{pr}^{(l)}\Delta = \iota_l^{l+l'} \circ \mathbf{pr}^{(l+l')}\Delta.$$

Corollary 4.5. *If Φ: $Z \to Z'$ is a smooth diffeomorphism, then there is a unique smooth diffeomorphism*

$$\mathbf{pr}^{(k)}\Phi: J_k^*(Z, p) \to J_k^*(Z', p),$$

called the k-th prolongation of Φ, such that for any parametrization f: $\mathbf{R}^p \to Z$ of a p-section of Z, the following diagram commutes:



Moreover

$$\mathbf{pr}^{(k')}\mathbf{pr}^{(k)}\Phi = \iota_k^{k+k'} \circ \mathbf{pr}^{(k+k')}\Phi.$$

The last corollary will be especially important when the prolongation of transformation group actions to extended jet bundles is discussed in §5.

We now proceed to describe the concept of a system of partial differential equations for an extended jet bundle. Recall that in the category of vector bundles a differential equation is usually given as $\Delta^{-1}\{0\}$ where Δ is a differential operator and 0 denotes the zero cross-section of the image bundle. In the case of fiber bundles, there is no such intrinsically defined cross-section. (Indeed, the bundle may have no smooth global sections.) Even if we restricted our attention to bundles with a distinguished cross-section, there would be problems in relating the prolonged differential equation to the prolonged differential operator. In view of these observations, the following definition can be seen to give an appropriate generalization of the notion of a system of partial differential equations.

Definition 4.6. Let $\Delta: J_k^*(Z,p) \to F$ be a smooth differential operator, and let $F_0 \subset F$ be a subbundle over Z. Then the *differential equation* corresponding to F_0 is the subvariety $\Delta^{-1}\{F_0\} \subset J_k^*(Z,p)$.

Note that if $\Delta_0 = \Delta^{-1}\{0\} \subset J_k^*(Z,p)$ is a subvariety of $J_k^*(Z,p)$ given by the zero set of some smooth map $\Delta: J_k^*(Z,p) \to \mathbf{R}^\alpha$, then Δ_0 is the differential equation corresponding to the zero cross-section of the trivial bundle $Z \times \mathbf{R}^\alpha$ under the differential operator $\pi_0^k \times \Delta$. Therefore any closed subvariety of $J_k^*(Z,p)$ can be viewed as the differential equation corresponding to some smooth differential operator. Suppose

$$\Delta^i(x, u, u^{(k)}) = 0, \ i = 1, \cdots, \alpha,$$

is a system of partial differential equations in some coordinate system on Z. Then the closure of the subset of $J_k^*(Z,p)$ given by these equations (which are only defined on an appropriate trivialized jet bundle) will be the differential equation corresponding to this system. Thus all classical systems of partial differential equations are included in our definition, with the added feature that solutions with "vertical tangents" are allowed, provided these tangents are in some sense the limits of "tangents" which satisfy the system of equations. In general, the interesting object will be the subvariety in the extended jet bundle and not the particular differential operator used to define it. Therefore a k-th order system of partial differential equations over a smooth manifold Z will be taken to mean an arbitrary closed subset $\Delta_0 \subset J_k^*(Z,p)$. A solution to a system of equations is a p-section $s \subset Z$ satisfying $j_k^* s \subset \Delta_0$.

Given a subset $S \subset Z$, let $J_k^*(S,p) \subset J_k^*(Z,p)$ denote the subset of all extended k-jets of p-sections of Z wholly contained in S. Note that this set will be empty if S contains no p-dimensional submanifolds.

Definition 4.7. Let $\Delta_0 \subset J_k^*(Z,p)$ be a k-th order system of partial differential equations. Then the *l-th prolongation* of Δ_0 is the $(k + l)$-th order differential equation

$$\mathbf{pr}^{(l)}\Delta_0 = J_l^*(\Delta_0, p) \cap J_{k+l}^*(Z, p),$$

where $J_{k+l}^*(Z, p) \subset J_l^*(J_k^*(Z, p), p)$ via the injection given in Corollary 4.3.

The next proposition shows that $\mathbf{pr}^{(l)}\Delta_0$ is indeed a differential equation and corresponds to the prolongation of the differential operator defining Δ_0, providing that this operator is in some sense "irreducible".

Proposition 4.8. *Let Δ: $J_k^*(Z, p) \to F$ be a smooth differential operator, and let $F_0 \subset F$ a subbundle such that Δ is transversal to F_0. Let $\Delta_0 = \Delta^{-1}\{F_0\}$ be the differential equation corresponding to F_0. Then*

$$\mathbf{pr}^{(l)}\Delta_0 = (\mathbf{pr}^{(l)}\Delta)^{-1}[J_l^*(F_0, p)].$$

Recall that a smooth map f: $M \to N$ is transversal to a submanifold $N_0 \subset N$ if for any $y \in f(M) \cap N_0$ with $f(x) = y$, $TN|_y = TN_0|_y + df[TM|_x]$. The transversality of Δ of F_0 is necessary for the proposition to hold. For instance, if the differential equation $u_x = 0$ is defined by the operator $\Delta = u_x^2$, then $\mathbf{pr}^{(1)}\Delta = (u_x^2, 2u_x u_{xx})$, so the equation in the second jet bundle given by $\mathbf{pr}^{(1)}\Delta$ is just $u_x = 0$, which is not the prolonged equation–$u_x = 0 = u_{xx}$. It can be seen that for polynomial operators, transversality is related to irreducibility.

Given a fiber bundle F over Z, a p-section of F will be said to be vertical at a point if its tangent space at that point has nonzero intersection with the tangent space to the fiber of F. Define $V_l^k \subset J_l^*(J_k^*(Z, p), p)$ to be the subset given by all l-jets of vertical sections of $J_k^*(Z, p)$.

Lemma 4.9. *For each k, l,*

$$J_{k+l}^*(Z, p) \cap V_l^k = \varnothing.$$

Lemma 4.10. *If s is a smooth p-section of $J_k^*(Z, p)$ such that $j_l^* s|_j \in J_{k+l}^*(Z, p)$ for some $j \in J_k^*(Z, p)$, then for any smooth differential operator Δ: $J_k^*(Z, p) \to F$,*

$$\mathbf{pr}^{(l)}\Delta(j_l^* s|_j) = j_l^*(\Delta \circ s)|_{\Delta(j)}.$$

Lemma 4.11. *Suppose F: $Z \to Z'$ is a smooth map between manifolds, and $Z_0' \subset Z'$ a submanifold transversal to F. Given $z \in Z$ with $F(z) = z' \in Z_0'$ and a p-section $s \in C^\infty(Z, p)|_z$ such that*

$$d^l F[\mathfrak{T}_l s|_z] \subset \mathfrak{T}_l Z_0'|_{z'},$$

then there exists another p-section $\hat{s} \in C^\infty(Z, p)|_z$ with $\mathfrak{T}_l \hat{s}|_z = \mathfrak{T}_l s|_z$ and $F(\hat{s}) \subset Z_0'$.

Proof. Choose local coordinates (x, y, t) around z such that $s = \{y = 0, t = 0\}$ and $dF(\partial/\partial y^i)$ form a basis for $TZ'|_{z'}/TZ_0'|_{z'}$. Choose local coordinates (ξ, η) around z' such that $Z_0' = \{\eta = 0\}$. Let

$$F(x, y, t) = (F_1(x, y, t), F_2(x, y, t))$$

in these coordinates, and, using the implicit function theorem, let $y = y(x)$ be the smooth solution to the equation $F_2(x, y(x), 0) = 0$ near $z = (0, 0, 0)$. Then $\hat{s} = \{(x, y(x), 0)\}$ satisfies the criteria of the lemma. Indeed, differentiating the equation which implicitly defines $y(x)$ gives

$$\frac{\partial^{\Sigma K}}{\partial x^K} F_2(0, 0, 0) + \frac{\partial^{\Sigma K}}{\partial x^K} y^j(x) \cdot \frac{\partial}{\partial y^j} F_2(0, 0, 0) + A_K = 0,$$

where A_K is a sum of terms involving derivatives of the $y^j(x)$ of orders $< \Sigma K$. Hence by induction all derivatives of $y(x)$ up to and including k-th order vanish.

Proof of Proposition **4.8.** Let $\Delta_0^{(l)} = [\mathbf{pr}^{(l)}\Delta]^{-1}\{J_l^*(F_0, p)\}$. To show that $\Delta_0^{(l)} \supset \mathbf{pr}^{(l)}\Delta_0$ let s be a p-section contained in Δ_0 with $j_l^* s \cap V_l^k = \varnothing$. This means that (locally) $\pi_0^k[s]$ is a smooth p-section of Z, hence $\Delta[s]$ is a smooth p-section of F_0 since $\rho \circ \Delta[s] = \pi_0^k[s]$. Therefore $j_l^*\Delta[s] \subset J_l^*(F_0, p)$. In particular, if $j_l^* s|_{j_0} \in J_{k+l}^*(Z, p)$, then by Lemma 4.10 we have

$$\mathbf{pr}^{(l)}\Delta(j_l^* s|_{j_0}) = j_l^*(\Delta[s])|_{\Delta(j_0)} \in J_l^*(F_0, p).$$

Now Lemma 4.9 implies $\Delta_0^{(l)} \supset \mathbf{pr}^{(l)}\Delta_0$.

Conversely, suppose $j \in \Delta_0^{(l)}$ and let $\pi_k^{k+l}(j) = j_0$ and $\pi_0^{k+l}(j) = z$. Let $s \in C^\infty(Z, p)|_z$ represent j so $\mathbf{pr}^{(l)}\Delta(j) = j_l^*(\Delta[j_k^* s])|_{\Delta(j_0)}$, hence $d^l\Delta[\mathfrak{T}_l(j_k^* s)|_{j_0}] \subset \mathfrak{T}_l F_0$. By Lemma 4.11 there exists $\hat{s} \in C^\infty(J_k^*(Z, p), p)|_{j_0}$ with $\mathfrak{T}_l \hat{s}|_{j_0} = \mathfrak{T}_l s|_{j_0}$ and $\hat{s} \subset \Delta_0$. In addition $j_l^* \hat{s}|_{j_0} = j$, hence $j \in \mathbf{pr}^{(l)}\Delta_0$.

5. Prolongation of group actions

Let Z be a smooth manifold of dimension $p + q$, and suppose that G is a smooth local group of transformations acting on Z. (The relevant definitions and results may be found in Palais' monograph [19].) Basically this means that there is an open subset U_0 with $\{e\} \times Z \subset U_0 \subset G \times Z$ and a smooth map Φ: $U_0 \to Z$ which is consistent with the local group structure of G. Define

$$Z_g = \{z \in Z: (g, z) \in U_0\} \quad \text{for } g \in G,$$

$$G_z = \{g \in G: (g, z) \in U_0\} \quad \text{for } z \in Z.$$

Let $\mathfrak{g} \subset TZ$ denote the involutive "quasi-differential system" spanned by the infinitesimal generators of the actions of one-parameter subgroups of G. (The adjective "quasi-" is used since the dimension of $\mathfrak{g}|_z$ may vary depending on z. However, since the subsets of Z where $\dim \mathfrak{g}|_z$ is constant are invariant under G, it may be assumed without significant loss of generality that all the orbits of G have the same dimension.)

Theorem 5.1. *A submanifold $S \subset Z$ is locally invariant under the group action of G iff $TS|_z \supset \mathfrak{g}|_z$ for all $z \in S$. A closed submanifold $S \subset Z$ is invariant iff it is locally invariant.*

Usually we shall be interested in the local invariance of subvarieties of Z which are given by the vanishing of a smooth function $F: Z \to \mathbf{R}^k$. Recall that F is a submersion if dF has maximal rank everywhere.

Theorem 5.2. *Let $S = F^{-1}\{0\}$ for $F: Z \to \mathbf{R}^k$ a smooth submersion. Then S is invariant under G iff*

$$dF[\mathfrak{g}|_z] = 0$$

for all $z \in S$.

This theorem is usually what is meant by the infinitesimal criterion for the invariance of a subvariety. In local coordinates $(z^1, \cdots, z^n)$ on Z, suppose $\mathfrak{g}$ is spanned by the vector fields

$$\mathbf{v}_i = \sum_{j=1}^{n} \xi_i^j(z)\frac{\partial}{\partial z^j}, \quad i = 1, \cdots, l.$$

If $F(z) = (F^1(z), \cdots, F^k(z))$, then $S = F^{-1}\{0\}$ is invariant under G iff

$$\mathbf{v}_i F^j = \sum \xi_i^k(z)\frac{\partial F^j}{\partial z^k}(z) = 0, \ i = 1, \cdots, l, \ j = 1, \cdots, k,$$

whenever $F^1(z) = \cdots = F^k(z) = 0$.

Given a local group of transformations G acting on Z, Corollary 4.5 shows that there is an induced action of G on the extended jet bundle $J_k^*(Z, p)$, which will be called the prolonged action of G. The construction of this prolonged action shows that if Δ_0 is a system of partial differential equations which is invariant under the prolonged action of G, then the elements of G transform solutions of Δ_0 to other solutions of Δ_0. The infinitesimal criterion of Theorem 5.2 will provide a practically useful method of determining when a system of partial differential equations is actually invariant under a prolonged group action. To implement this, we need to find a local coordinate expression for the prolongation of the infinitesimal generators of G. Note that it suffices to consider the case when G is a one-parameter group with infinitesimal generator given by a vector field $\mathbf{v}$.

Definition 5.3. Given a vector field $\mathbf{v}$ on Z, let $\exp(t\mathbf{v})$ denote the local one-parameter group generated by $\mathbf{v}$. The *k-th prolongation* of $\mathbf{v}$ is the vector field on $J_k^*(Z, p)$ given by

$$\mathbf{pr}^{(k)}\mathbf{v}|_j = \frac{d}{dt}\bigg|_{t=0} \mathbf{pr}^{(k)}[\exp(t\mathbf{v})](j), \ j \in J_k^*(Z, p).$$

The next theorem provides the explicit local coordinate expression for $\mathbf{pr}^{(k)}\mathbf{v}$. The coordinate charts on $J_k^*(Z, p)$ are those provided by Proposition 3.14.

Theorem 5.4. *Let* $\chi\colon Z \to \mathbf{R}^p \times \mathbf{R}^q$ *be a local coordinate chart on* Z *with induced coordinates*

$$\chi^{(k)}\colon J_k^*(Z, p; \mathcal{U}) \to \mathbf{R}^p \times \mathbf{R}^q \times \odot_*^k(\mathbf{R}^p, \mathbf{R}^q)$$

on the extended jet bundle. Let $\mathbf{v}$ *be a smooth vector field on* Z *given in these local coordinates by*

$$\mathbf{v} = \xi^1(x, u)\frac{\partial}{\partial x^1} + \cdots + \xi^p(x, u)\frac{\partial}{\partial x^p} + \phi_1(x, u)\frac{\partial}{\partial u^1} + \cdots + \phi_q(x, u)\frac{\partial}{\partial u^q},$$

where $(x, u) = (x^1, \cdots, x^p, u^1, \cdots, u^q)$ *are the coordinates on* $\mathbf{R}^p \times \mathbf{R}^q$. *Then in the induced coordinates* $(x, u, \cdots, u_J^i, \cdots)$ *we have*

$$\mathbf{pr}^{(k)}\mathbf{v} = \mathbf{pr}^{(k-1)}\mathbf{v} + \sum_{J \in \mathbb{S}_k^p} \phi_i^J(x, u, \cdots, u_K^i, \cdots)\frac{\partial}{\partial u_J^i},$$

where

$$\phi_i^J = D_J\left(\phi_i - \sum_{\sigma=1}^{p} u_\sigma^i \xi^\sigma\right), \quad u_\sigma^i = \partial u^i/\partial x^\sigma, \tag{5.1}$$

and D_J *is the total derivative from Lemma* 2.7.

Proof. Let $\hat{\Phi}_t\colon Z \to Z$ denote the local transformation $\exp(t\mathbf{v})$. In local coordinates, for t sufficiently small,

$$\hat{\Phi}_t(x, u) = (\Phi_t(x, u), \Psi_t(x, u))$$

with

$$\left.\frac{d}{dt}\right|_{t=0} \Phi_t^j(x, u) = \xi^j(x, u), \qquad j = 1, \cdots, p,$$

$$\left.\frac{d}{dt}\right|_{t=0} \Psi_t^i(x, u) = \phi_i(x, u), \qquad i = 1, \cdots, q.$$

Let $j = (z, u^{(k)}) = (x, u, \cdots, u_J^i, \cdots) \in J_k^*(Z, p; \mathcal{U})$, and let $\hat{f}\colon \mathbf{R}^p \to Z$ be the normal parametrization of a section representing j. In local coordinates, $\hat{f}(x) = (x, f(x))$ and $u^{(k)} = \partial_*^k f(x)$. Let $\hat{g}_t = \hat{\Phi}_t \circ \hat{f}$ for t sufficiently small so that the sections im $\hat{g}_t$ are transversal to $\mathcal{U}$. Let $\hat{f}_t\colon \mathbf{R}^p \to Z$, $\hat{f}_t(x) = (x, f_t(x))$, be the renormalized parametrization of im $\hat{g}_t$. It follows that

$$f_t = \Psi_t \circ \hat{f} \circ \left(\Phi_t \circ \hat{f}\right)^{-1}$$

for t sufficiently small so the inverse exists. Therefore

$$\mathbf{pr}^{(k)}\hat{\Phi}_t(z, u^{(k)}) = \left(\hat{\Phi}_t(z), \partial_*^k\left[\Psi_t \circ \hat{f} \circ \left(\Phi_t \circ \hat{f}\right)^{-1}\right](x)\right).$$

We must compute the derivative of this expression with respect to t at $t = 0$.

Making use of the fact that

$$\frac{d}{dt}A^{-1}(t) = -A^{-1}(t)\cdot\frac{d}{dt}A(t)\cdot A^{-1}(t)$$

for any differentiable matrix valued function of t, we have

$$(*)\qquad \begin{aligned}\mathbf{pr}^{(k)}\mathbf{v}|_j &= \frac{d}{dt}\bigg|_{t=0}\mathbf{pr}^{(k)}\hat{\Phi}_t(z, u^{(k)})\\ &= \mathbf{v} + \frac{d}{dt}\bigg|_{t=0}\partial_*^k(\Psi_t\circ f) - \partial_*^k f\cdot\frac{d}{dt}\bigg|_{t=0} d^k(\Phi_t\circ f),\end{aligned}$$

since $\Psi_0\circ\hat{f} = f$ and $\Phi_0\circ\hat{f} = \mathbf{1}_p$. The second term in $(*)$ is just the total derivative matrix $D_*^k\phi(x, u, u^{(k)})$ of $\phi = (\phi_1, \cdots, \phi_q)$ since the entries of $\partial_*^k(\Psi_t\circ\hat{f})$ are just the various partial derivatives $\partial_K(\Psi_t^i\circ\hat{f})$. We are thus allowed to interchange the order of differentiation:

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0}\partial_*^k(\Psi_t\circ\hat{f}) &= \partial_*^k\left(\frac{d}{dt}\bigg|_{t=0}\Psi_t\circ\hat{f}\right)\\ &= \partial_*^k(\phi\circ(\mathbf{1}_p\times f))\\ &= D_*^k\phi(x, u, u^{(k)}).\end{aligned}$$

The third term requires more careful analysis since the matrix entries do not depend linearly on functions of t. Let $(\partial_j^i g)$ be the block matrix form of $d^k g$ given in (2.6). We have

$$\left[\partial_*^k f\cdot\frac{d}{dt}\bigg|_{t=0} d^k(\Phi_t\circ\hat{f})\right]_m^1 = \sum_{i=1}^m \partial^i f\cdot\frac{d}{dt}\bigg|_{t=0}\partial_m^i(\Phi_t\circ\hat{f}).$$

Moreover

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0}\partial_m^i(\Phi_t\circ\hat{f}) &= \frac{d}{dt}\bigg|_{t=0}\sum_{J\in\mathcal{Q}_m^i}\frac{1}{J!}\partial(\Phi_t\circ\hat{f})^{j_1}\odot\cdots\odot\partial^k(\Phi_t\circ\hat{f})^{j_k}\\ &= \frac{\mathbf{1}_p^{i-1}}{(i-1)!}\odot D^{m-i+1}\xi(x, u, u^{(k)}),\end{aligned}$$

where $\xi = (\xi^1, \cdots, \xi^p)$. The proof of the last equality is a consequence of the fact that

$$\Phi_0\circ\hat{f} = \mathbf{1}_p, \text{ so } \partial(\Phi_0\circ\hat{f}) = \mathbf{1}_p \text{ and } \partial^2(\Phi_0\circ\hat{f}) = \cdots = \partial^k(\Phi_0\circ\hat{f}) = 0.$$

Therefore the only multi-indices in $\mathcal{Q}_m^i$ contributing anything to the sum upon evaluating the derivative at $t = 0$ are $(i-1)\delta^1 + \delta^{m-i+1}$ and $i\delta^1$. Recall that for $l \geqslant 1$,

$$\frac{1}{(l-1)!}\mathbf{1}_p^{l-1}$$

is just the identity map of $\odot_{l-1}\mathbf{R}^p$. Therefore, by (2.1),

$$(**)\qquad \begin{aligned}\left\langle e_J, \frac{\mathbf{1}_p^{l-1}}{(l-1)!}\odot D^{j-l+1}\xi\right\rangle &= \sum_{J>K\in\mathfrak{S}^p}\binom{J}{K}e_{J-K}\langle e_K, D^{j-l+1}\xi\rangle\\ &= \sum_{J>K}\sum_{\sigma=1}^{p}\binom{J}{K}D_K\xi^\sigma\cdot e_{J-K+\delta^\sigma}.\end{aligned}$$

Now the matrix entries of $(*)$ are

$$(***)\qquad \phi_i^J = D_J\phi_i - \sum u_L^i C_J^L.$$

The sum in $(***)$ is taken over all multi-indices L with $\Sigma L = l \leqslant j = \Sigma J$, and C_J^L is the coefficient of e_L in $(**)$. Leibnitz' formula for the derivatives of a product completes the proof of (5.1).

Example 5.5. Consider the special case $p = 2$, $q = 1$ with coordinates (x, y, u) and vector field $\xi\partial_x + \eta\partial_y + \phi\partial_u$. The second prolongation of this vector field is given by

$$\xi\partial_x + \eta\partial_y + \phi\partial_u + \phi^x\partial_{u_x} + \phi^y\partial_{u_y} + \phi^{xx}\partial_{u_{xx}} + \phi^{xy}\partial_{u_{xy}} + \phi^{yy}\partial_{u_{yy}},$$

where

$$(5.2)\qquad \begin{aligned}\phi^x &= D_x\phi - u_xD_x\xi - u_yD_x\eta\\ &= \phi_x + u_x(\phi_u - \xi_x) - u_x^2\xi_u - u_y\eta_x - u_xu_y\eta_u,\\ \phi^y &= D_y\phi - u_xD_y\xi - u_yD_y\eta\\ &= \phi_y + u_y(\phi_u - \eta_y) - u_y^2\eta_u - u_x\xi_y - u_xu_y\xi_u,\\ \phi^{xx} &= D_{xx}\phi - 2u_{xx}D_x\xi - 2u_{xy}D_x\eta - u_xD_{xx}\xi - u_yD_{xx}\eta,\\ \phi^{xy} &= D_{xy}\phi - u_{xx}D_y\xi - u_{xy}D_y\eta - u_{xy}D_x\xi - u_{yy}D_x\eta\\ &\quad - u_xD_{xy}\xi - u_yD_{xy}\eta,\\ \phi^{yy} &= D_{yy}\phi - 2u_{xy}D_y\xi - 2u_{yy}D_y\eta - u_xD_{yy}\xi - u_yD_{yy}\eta.\end{aligned}$$

The next corollary can be found in [8, p. 106] and provides a useful recursion formula for computing the functions ϕ_i^J.

Corollary 5.6. *The coefficient functions in Theorem 5.4 satisfy the recursion relation*

$$(5.3)\qquad \phi_i^{J+\delta^k} = D_k\phi_i^J - \sum_{\sigma=1}^{p} u_{J+\delta^\sigma}^i D_k\xi^\sigma.$$

Finally, we may use Theorem 5.4 to show that the prolongation operator preserves the Lie algebra structure of the space of vector fields. An alternate noncomputational proof of this fact may be found in reference [16].

Theorem 5.7. *Let* $\mathbf{v}$, $\mathbf{v}'$ *be smooth vector fields on* Z, *and let* a, b *be real constants. Then for any* $k > 0$,

$$\mathbf{pr}^{(k)}(a\mathbf{v} + b\mathbf{v}') = a \cdot \mathbf{pr}^{(k)}\mathbf{v} + b \cdot \mathbf{pr}^{(k)}\mathbf{v}',$$
$$\mathbf{pr}^{(k)}[\mathbf{v}, \mathbf{v}'] = [\mathbf{pr}^{(k)}\mathbf{v}, \mathbf{pr}^{(k)}\mathbf{v}'].$$

6. Symmetry groups of differential equations

Consider a k-th order system of partial differential equations $\Delta_0 \subset J_k^*(Z, p)$. The "symmetry group" of Δ_0 will be loosely taken to mean the local group of all smooth local transformations of Z whose k-th prolongation to $J_k^*(Z, p)$ leaves Δ_0 invariant. The algebra of infinitesimal symmetries of Δ_0 will be the space of smooth vector fields on Z whose k-th prolongations leave Δ_0 infinitesimally invariant. Note that by Lemma 5.7, the infnitesimal symmetries form a Lie algebra. In general, it is to be expected that the symmetry algebra exponentiates to form the connected component of the identity of the symmetry group. A technical problem arises in the case the symmetries form an infinite dimensional algebra: a Lie pseudogroup type condition (cf. [22]) must be imposed to maintain the correspondence between the group and the algebra. It shall be seen that the infinitesimal symmetries must satisfy a large number of partial differential equations so that under some appropriate weak conditions on Δ_0 the Lie pseudogroup criteria will be satisfied. However, as these are rather technical in nature and shed little additional light on the subject, they will not be investigated here. Besides, we will usually be concerned with finite dimensional subgroups of the symmetry group, and problems of this nature will not arise.

In practice, Δ_0 will not be given as a subvariety of $J_k^*(Z, p)$, but will be given, in local coordinates (x, u) on Z, as a system of equations

$$\Delta^i(x, u, u^{(k)}) = 0, \quad i = 1, \cdots, \alpha, \tag{6.1}$$

where the Δ^i's are smooth real valued functions on $J_k^*(Z, p; \mathfrak{U})$. Here $\mathfrak{U}$ denotes the differential system spanned by $\{\partial/\partial u^1, \cdots, \partial/\partial u^q\}$ and the $u^{(k)}$ are the induced coordinates on the trivialized jet bundle. In the classical case $Z = \mathbf{R}^p \times \mathbf{R}^q$ and (6.1) are given on $J_k(\mathbf{R}^p, \mathbf{R}^q) = J_k^*(Z, p; \mathfrak{U})$. Then Δ_0 will denote the closure of the subvariety of $J_k^*(Z, p; \mathfrak{U})$ given by (6.1) in $J_k^*(Z, p)$. Note that to check the invariance of Δ_0 it suffices to check the local invariance of the subvariety defined by (6.1), so we can effectively restrict our attention to the trivialized jet bundle.

Let Δ: $J_k^*(Z, p; \mathfrak{U}) \to \mathbf{R}^\alpha$ denote the map with components Δ^i. If Δ is a submersion, then the infinitesimal criterion of invariance shows that Δ_0 is invariant under the group G iff

$$\mathbf{pr}^{(k)}\mathbf{v}[\Delta] = 0 \text{ whenever } \Delta = 0 \tag{6.2}$$

for all infinitesimal generators $\mathbf{v}$ of G. In the case that Δ_0 is "irreducible," meaning that any real valued function f vanishing on Δ_0 must be of the form $f = \sum \lambda_i \Delta^i$, where the λ_i's are smooth real valued functions on $J_k^*(Z, p)$, then condition (6.2) becomes

$$\mathbf{pr}^{(k)}\mathbf{v}[\Delta] = \sum \lambda_i \Delta^i. \tag{6.3}$$

In practice (6.3) is the condition most frequently used–note that a priori the λ_i's can depend on all the derivatives as well as the dependent and independent variables. To calculate the symmetry group of such a system of equations, $\mathbf{v}$ is allowed to be an arbitrary vector field:

$$\mathbf{v} = \sum_1^p \xi^i(x, u)\frac{\partial}{\partial x^i} + \sum_1^q \phi_j(x, u)\frac{\partial}{\partial u^j},$$

where the ξ^i's and ϕ_j's are unknown functions of the variables (x, u). Using the prolongation formula (5.1) the vector field $\mathbf{pr}^{(k)}\mathbf{v}$ is computed in terms of the ξ^i, ϕ_j and their derivatives. Then condition 6.3 provides a large system of partial differential equations which these functions must satisfy, the general solution of which is the desired (infinitesimal) symmetry group.

Example 6.1 (*Burgers' equation*). Let $Z = \mathbf{R}^2 \times \mathbf{R}$ with coordinates (x, t, u) and consider the second order quasi-linear equation

$$u_t + uu_x + u_{xx} = 0, \tag{6.4}$$

known as Burgers' equation. It is important in nonlinear wave theory, being the simplest equation which contains both nonlinear propagation and diffusion. See, for instance, [25, Chapter 4] for a fairly complete discussion of its properties. Let $\mathbf{v} = \xi\partial_x + \tau\partial_t + \phi\partial_u$ be a smooth vector field on Z, with second prolongation

$$\mathbf{pr}^{(2)}\mathbf{v} = \mathbf{v} + \phi^x\partial_{u_x} + \phi^t\partial_{u_t} + \phi^{xx}\partial_{u_{xx}} + \phi^{xt}\partial_{u_{xt}} + \phi^{tt}\partial_{u_{tt}},$$

where the coefficient functions are given by (5.2).

Using Criterion 6.3, we have that

$$\phi^t + u\phi^x + u_x\phi + \phi^{xx} = \lambda(u_t + uu_x + u_{xx}) \tag{6.5}$$

must be satisfied for some function λ which might depend on $(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt})$. However, as the second order derivatives u_{xx}, u_{xt}, u_{tt} occur only linearly in ϕ^{xx}, λ can only depend on (x, t, u, u_x, u_t). The coefficient of u_{xt} in (6.5) is

$$-2D_x\tau = 0,$$

implying that τ is a function t alone. The coefficient of u_{xx} is

(6.6) $$\phi_u - 2\xi_x - 3u_x\xi_u = \lambda,$$

which determines λ. The coefficient of u_t is, since λ is independent of u_t,

$$\phi_u - \tau_t - u_x\xi_u = \lambda,$$

which implies that $\xi_u = 0$, $\tau_t = 2\xi_x$, so $\xi_{xx} = 0$ and λ depends only on (t, u). The coefficient of u_x^2 is now $\phi_{uu} = 0$, hence

$$\phi(x, t, u) = \alpha(x, t) + u\beta(x, t).$$

The coefficient of u_x is now

$$-\xi_t + u(\beta - \xi_x) + \alpha + \beta u + 2\beta_x = \lambda,$$

hence (6.6) shows

$$\beta = -\xi_x, \quad \beta_x = 0, \quad \alpha = \xi_t.$$

Finally, the terms in (6.5) not involving any derivatives of u are

$$\phi_t + u\phi_x + \phi_{xx} = 0.$$

Thus $\alpha_t = 0$, $\beta_t + \alpha_x = 0$, which implies $\xi_{tt} = 0$. Therefore the general solution to (6.5) is given by

$$\begin{aligned}
\xi &= c_1 + c_3x + c_4t + c_5xt,\\
\tau &= c_2 + 2c_3t + c_5t^2,\\
\phi &= c_4 + c_5x - (c_3 + c_5t)u,\\
\lambda &= -3c_3 - 3c_5t,
\end{aligned}$$

where c_1, c_2, c_3, c_4, c_5 are arbitrary real constants. The infinitesimal symmetry algebra of Burgers' equation is five dimensional with basis consisting of the vector fields

(6.7) $$\begin{aligned}
\mathbf{v}_1 &= \partial_x,\\
\mathbf{v}_2 &= \partial_t,\\
\mathbf{v}_3 &= x\partial_x + 2t\partial_t - u\partial_u,\\
\mathbf{v}_4 &= t\partial_x + \partial_u,\\
\mathbf{v}_5 &= xt\partial_x + t^2\partial_t + (x - tu)\partial_u.
\end{aligned}$$

The commutation relations between these vector fields is given by the following table, the entry at row i and column j representing $[\mathbf{v}_j, \mathbf{v}_j]$:

(6.8)

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$
$\mathbf{v}_1$	0	0	$\mathbf{v}_1$	0	$\mathbf{v}_4$
$\mathbf{v}_2$	0	0	$2\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_3$
$\mathbf{v}_3$	$-\mathbf{v}_1$	$-2\mathbf{v}_2$	0	$\mathbf{v}_4$	$2\mathbf{v}_5$
$\mathbf{v}_4$	0	$-\mathbf{v}_1$	$-\mathbf{v}_4$	0	0
$\mathbf{v}_5$	$-\mathbf{v}_4$	$-\mathbf{v}_3$	$-2\mathbf{v}_5$	0	0

Let G_i denote the one-parameter local group associated with $\mathbf{v}_i$. Then G_1 and G_2 are just translations in the x and t directions respectively:

(6.9a)
$$G_1\colon (x, t, u) \mapsto (x + \lambda, t, u),$$
$$G_2\colon (x, t, u) \mapsto (x, t + \lambda, u), \ \lambda \in \mathbf{R},$$

and represent the fact that Burgers' equation has no dependence on x or t. The group G_3 consists of scale transformations:

(6.9b)
$$G_3\colon (x, t, u) \to (e^{\lambda}x, e^{2\lambda}t, e^{-\lambda}u), \ \lambda \in \mathbf{R}.$$

G_4 is a group of Gallilean type:

(6.9c)
$$G_4 : (x, t, u) \to (x + \lambda t, t, u + \lambda),$$
$$G_5 : (x, t, u) \to \left(\frac{t}{1 - \lambda t}, \frac{x}{1 - \lambda t}, u + \lambda(x - tu)\right), \ \lambda \in \mathbf{R}.$$

These will be discussed in more detail in Example 8.10.

7. Groups of equivalent systems

Given an n-th order partial differential equation, there is a standard trick used to convert it into an equivalent system of first order partial differential equations. For instance, in the case of the heat equation $u_t = u_{xx}$, the equivalent system is

(7.1)
$$u_x = v, \ u_t = w, \ v_t = w_x, \ v_x = w.$$

In the author's thesis [16], the symmetry groups of both the equation $u_t = u_{xx}$ and the system (7.1) were computed, and it was shown that the symmetry group of the first order system can be viewed the first prolongation of the symmetry group of the second order equation. It is the aim of this section to investigate in what sense this phenomenon is true in general. It will be shown that, barring the presence of "higher order symmetries," the symmetry group of a first order system is the prolongation of the symmetry group of an equivalent higher order equation (at least locally). The higher order symmetries are groups whose transformations depend on the derivatives as well as just the independent and dependent variables in the equation. At the end of this section, an example of an equation which possesses higher order symmetries–the wave equation–will be considered.

We first need to describe what exactly is meant by the equivalent first order system of a partial differential equation in the language of extended jet bundles. Suppose $\Delta_0 \subset J_k^*(Z, p)$ is a k-th order system of partial differential equations. If $(u^1, \cdots, u^q)$ denote the dependent variables corresponding to some coordinate system (x, u) on Z, then the dependent variables in the equivalent first order system will be the induced coordinates u_K^i for all multi-indices $K \in \mathbb{S}^p$ with $0 \leqslant \Sigma K < k$. (Here we identify u_0^i with u^i.) In other words, we are considering Δ_0 as a first order system of equations over the new manifold $J_{k-1}^*(Z, p)$, i.e., as a subvariety of $J_1^*(J_{k-1}^*(Z, p), p)$. Using the embedding ι_{k-1}^k given in Corollary 4.3 it is not hard to see that this first order system is nothing but

$$\iota_{k-1}^k(\Delta_0) \subset J_1^*(J_{k-1}^*(Z, p), p).$$

Example 7.1. Consider the manifold $Z = \mathbf{R}^2 \times \mathbf{R}$ with coordinates (x, t, u), and let $k = 2$. Suppose $\Delta_0 \subset J_2^*(Z, 2)$ is given by the equations

$$\Delta^i(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0, \quad i = 1, \cdots, \alpha.$$

The local coordinates of $J_1^*(Z, 2)$ will be denoted by (x, t, u, v, w), where v corresponds to u_x and w to u_t. Then the local coordinates on $J_1^*(J_1^*(Z, 2), 2)$ are $(x, t, u, v, w, u_x, u_t, v_x, v_t, w_x, w_t)$, and the subbundle $J_2^*(Z, 2)$ (which we will henceforth identify with $\iota_1^2(J_2^*(Z, 2))$) is given by the equations

$$u_x = v, \quad u_t = w, \quad v_t = w_x. \tag{7.2}$$

Therefore the first order system corresponding to Δ_0 is given by (7.2) and the additional equations

$$\Delta^i(x, t, u, v, w, v_x, w_x, w_t) = 0, \quad i = 1, \cdots, \alpha.$$

More generally, we can replace a system of k-th order partial differential equations by an equivalent system of $(k - l)$th order equations for some $1 \leqslant l < k$. This is accomplished via the embedding ι_l^k from Corollary 4.3:

$$\Delta_0 \subset J_k^*(Z, p) \subset J_{k-l}^*(J_l^*(Z, p), p).$$

Lemma 7.2. *Let $\Delta_0 \subset J_k^*(Z, p)$ be a k-th order system of partial differential equations. Let $1 \leqslant l < k$, and suppose $s' \subset J_l^*(Z, p)$ is a p-section such that $j_{k-l}^* s' \subset \Delta_0$. Then locally $s' = j_l^* s$ for some p-section $s \subset Z$ which is a solution to Δ_0.*

Proof. Let $s = \pi_0^l(s')$. Since $j_{k-l}^* s' \subset J_k^*(Z, p)$, Lemma 4.9 shows that locally s is a p-section of Z. The local coordinate description of $j_{k-l}^* s'$ demonstrates that $s' = j_l^* s$, which proves the result.

Note that the projection $\pi_0^l(s')$ is not necessarily a global p-section of Z since there might be self-intersections. This lemma justifies the use of the word "equivalent", since the smooth solutions of the higher and lower order equivalent systems are in one-to-one correspondence via the extended jet map. Now we are in a position to consider the symmetry groups of the equivalent systems. Suppose G is a local group of transformations acting on Z whose k-th prolongation leaves Δ_0 invariant. Corollary 4.4 demonstrates that $J_k^*(Z, p)$ is an invariant subvariety of $\mathbf{pr}^{(k-l)}[\mathbf{pr}^{(l)}G]$ acting on $J_{k-l}^*(J_l^*(Z, p), p)$. Moreover

$$\mathbf{pr}^{(k-l)}\left[\mathbf{pr}^{(l)}G\right]|J_k^*(Z, p) = \mathbf{pr}^{(k)}G.$$

We conclude that if G is a symmetry group of a k-th order system of partial differential equations, then $\mathbf{pr}^{(l)}G$ is a symmetry group of the equivalent $(k - l)$-th order system.

Conversely, suppose G' is a local group of transformations acting on $J_l^*(Z, p)$ such that Δ_0 is an invariant subvariety of the prolonged action $\mathbf{pr}^{(k-l)}G'$. An obvious necessary condition for G' to satisfy in order to be the prolongation of some group G acting on Z is that it be *projectable*, i.e., if $\pi_0^l(j) = \pi_0^l(j')$ for $j, j' \in J_l^*(Z, p)$, then $\pi_0^l(gj) = \pi_0^l(gj')$ for all $g \in G_j' \cap G_{j'}'$. The non-projectable groups will be called *higher order symmetries*, since while they do transform solutions of Δ_0 to solutions of Δ_0, the transformations depend on the derivatives of the solutions as well as the solution values themselves. They can be considered a special case of nonpoint transformations; cf. [18].

A second criterion that the group G' must meet in order to be a prolongation is that Δ_0 is really a k-th order equation. For instance, if $\Delta_0 = (\pi_l^k)^{-1}[\Delta_0']$ for some l-th order equation Δ_0', then any transformation of $J_l^*(Z, p)$ leaving Δ_0' invariant will preserve Δ_0, and the projectable ones are not expected to necessarily be prolonged group actions. What can be proven is summarized in the next theorem.

Theorem 7.3. *Let $1 \leqslant l < k$ be integers, and let*

$$\Delta_0 \subset J_k^*(Z, p) \subset J_{k-l}^*(J_l^*(Z, p), p)$$

be a k-th order system of partial differential equations. If G is a local group of transformations acting on Z such that Δ_0 is invariant under the prolonged group action $\mathbf{pr}^{(k)}G$, then Δ_0 is invariant under $\mathbf{pr}^{(k-l)}[\mathbf{pr}^{(l)}G]$. Conversely, let G be a local group of transformations acting projectably on $J_l^*(Z, p)$ with projected group action G on Z. If Δ_0 is invariant under $\mathbf{pr}^{(k-l)}G'$, then $\mathbf{pr}^{(l)}G$ agrees locally with G' on $\pi_l^k[\Delta_0]$, which is a G' invariant subvarity of $J_l^*(Z, p)$.

Proof. The first result has already been demonstrated in the remarks preceding the statement of the theorem. To prove the converse, note that it suffices to show $\mathfrak{g}'$, the algebra of infinitesimal generators of G', agrees with $\mathbf{pr}^{(l)}\mathfrak{g}$, the l-th prolongation of the algebra of infinitesimal generators of G. To do this, it suffices to check that if $\mathbf{v}'$ is any projectable vector field on $J_l^*(Z, p)$ with projection $\mathbf{v}$ on Z such that Δ_0 is invariant under $\mathbf{pr}^{(k-l)}\mathbf{v}'$, then $\mathbf{pr}^{(l)}\mathbf{v} = \mathbf{v}'$ on $\pi_l^k[\Delta_0]$.

Choose local coordinates (x, u) on Z with induced local coordinates $(x, u, u^{(k)})$ on $J_k^*(Z, p)$ and $(x, u, u^{(l)})$ on $J_l^*(Z, p)$ so that $\pi_l^k(x, u, u^{(k)}) = (x, u, u^{(l)})$. The induced local coordinates on $J_{k-l}^*(J_l^*(Z, p), p)$ are given by $(x, u, u^{(l)}, u^{(k-l)}, (u^{(l)})^{(k-l)})$, the individual matrix entries given by $(u_J^i)_K$ for all $1 \leqslant i \leqslant q$, $J, K \in \mathbb{S}^p$, with $0 \leqslant \Sigma J \leqslant l$, $0 \leqslant \Sigma K \leqslant k - l$, where as usual we identify u_0^i with u^i and $(u_J^i)_0$ with u_J^i. By Corollary 4.3, $J_k^*(Z, p)$ is given by the equations

$$(u_J^i)_K = (u_{J'}^i)_{K'}, \quad i = 1, \cdots, q, \quad J + K = J' + K'.$$

Now let $\mathbf{v}'$ be given in local coordinates by

$$\mathbf{v}' = \sum \xi^j \frac{\partial}{\partial x^j} + \sum \phi_i \frac{\partial}{\partial u^i} + \sum \phi_i^J \frac{\partial}{\partial u_J^i},$$

so that

$$\mathbf{v} = \sum \xi^j \frac{\partial}{\partial x^j} + \sum \phi_i \frac{\partial}{\partial u^i},$$

and

$$\mathbf{pr}^{(k-l)}\mathbf{v}' = \mathbf{v}' + \sum (\phi_i^J)^K \frac{\partial}{\partial (u_J^i)_K}.$$

The $(\phi_i^J)^K$ are given by the prolongation formula. Note that since $\mathbf{v}'$ is projectable, ξ^j and ϕ_i are functions of (x, u) only.

Applying the infinitesimal criterion of invariance to Δ_0 considered as a $(k - l)$-th order system, we must have

$$(\phi_i^J)^K(\hat{j}) = (\phi_i^{J'})^{K'}(\hat{j}), \quad i = 1, \cdots, q, \quad J + K = J' + K',$$

whenever $\hat{j} \in \Delta_0$. In particular, let $K = \delta^\sigma$, $K' = 0$ for some $1 \leqslant \sigma \leqslant p$. The prolongation formula implies that

$$D_\sigma \phi_i^J - \sum_\tau (u_J^i)_\tau D_\sigma \xi^\tau = \phi_i^{J+\delta^\sigma}$$

on Δ_0 for all $0 \leqslant \Sigma J \leqslant l - 1$. Note that these are precisely the recursion relations for the prolongation of vector fields as given in Corollary 5.6 since $\Delta_0 \subset J_k^*(Z, p)$.

Note that the theorem does not imply that G' and $\mathbf{pr}^{(l)}G$ agree everywhere on $J_l^*(Z, p)$. For instance, in the coordinates of Example 7.1, an equation might be invariant under the transformation $(x, t, u, v, w) \to (x, t, u, w, v)$, but this projects to the identity transformation on $Z = \mathbf{R}^2 \times \mathbf{R}$, so is not a prolongation.

Example 7.4 (*The wave equation*). Let $Z = \mathbf{R}^2 \times \mathbf{R}$ with coordinates (x, t, u), and consider the second order equation $\Delta_0 \subset J_2^*(Z, 2)$ given by

$$u_{tt} = u_{xx}. \tag{7.3}$$

The equivalent first order system for Δ_0 is given by

$$u_x = v, \quad u_t = w, \quad v_t = w_x, \quad v_x = w_t, \tag{7.4}$$

where (x, t, u, v, w) are local coordinates on $J_1^*(Z, 2)$. Through some tedious calculations similar to those in Example 6.1, we derive the fact that the infinitesimal symmetry algebra of (7.4) is the space of all vector fields $\xi\partial_x + \tau\partial_t + \phi\partial_u + \psi\partial_v + \chi\partial_w$ with

$$\begin{aligned}
\xi &= f_1(v + w, x + t) + g_1(v - w, x - t),\\
\tau &= f_1(v + w, x + t) - g_1(v - w, x - t),\\
\phi &= (v + w)f_1(v + w, x + t) - f(v + w, x + t)\\
&\quad + (v - w)g_1(v - w, x - t) - g(v - w, x - t)\\
&\quad + cu + a(v + w) + b(v - w),\\
\psi &= -f_2(v + w, x + t) - g_2(v - w, x - t) + cv,\\
\chi &= -f_2(v + w, x + t) + g_2(v - w, x - t) + cw,
\end{aligned} \tag{7.5}$$

where f and g are arbitrary functions of two variables (the subscripts indicating partial derivatives with respect to the variables), a and b arbitrary functions of a single variable, and c an arbitrary constant.

To see what is going on with the higher order symmetries, let us consider a specific example. Let

$$f(\alpha, \eta) = \tfrac{1}{4}\alpha^2, \; g(\beta, \zeta) = \tfrac{1}{4}\beta^2, \; a = b = c = 0,$$

so the vector field under consideration is

$$\mathbf{v}_0' = v\partial_x + w\partial_t + \tfrac{1}{2}(v^2 + w^2)\partial_u.$$

The one-parameter group generated by v_0' is

$$G_0': (x, t, u, v, w) \to \left(x + \lambda v, t + \lambda w, u + \frac{\lambda}{2}(v^2 + w^2), v, w\right), \; \lambda \in \mathbf{R}.$$

Suppose $u = F(x, t)$ is a solution to the wave equation (7.3). Invariance of (7.4) under $\mathbf{pr}^{(1)}G_0'$ implies that if we solve the implicit equations

$$v = F_x(x + \lambda v, t + \lambda w), \; w = F_t(x + \lambda v, t + \lambda w)$$

for v, w, then

$$u = F(x + \lambda v, t + \lambda w) - \frac{\lambda}{2}(v^2 + w^2)$$

gives a one-parameter family of solutions. This may be verified directly by taking derivatives.

8. Group invariant solutions

Again suppose that Z is a smooth manifold of dimension $p + q$, and G is a local group of transformations acting on Z. Under some mild regularity conditions on the action of G, the quotient space Z/G can be endowed with the structure of a smooth manifold. Suppose further that Δ_0 is a system of partial differential equations in p independent variables on Z, i.e., a closed subset of $J_k^*(Z, p)$, which is invariant under the prolonged action of G. The fundamental theorem concerning G-invariant solutions to Δ_0 is that there is a system of partial differential equations $\Delta_0/G \subset J_k^*(Z/G, p - l)$, where l is the dimension of the orbits of G, whose solutions are in one-to-one correspondence with the G-invariant solutions of Δ_0. The local form of this general result is essentially due to Ovsjannikov [17], although special cases may be found in the work of Birkhoff and Sedov on dimensional analysis; cf. [2] and [21]. Morgan [15] gives another early version of this theorem. Note that the important point is that the number of independent variables in the new system of equations Δ_0/G is l fewer than those in Δ_0, making it in some sense easier to solve. The practical application of this method will be illustrated by the example of Burgers' equation at the end of this section.

Definition 8.1. An orbit $\mathcal{O}$ of G is said to be *regular* if for each $z \in \mathcal{O}$ there exist arbitrarily small neighborhoods V containing z with the property that for any orbit $\mathcal{O}'$ of G, $\mathcal{O}' \cap V$ is pathwise connected. The group G *acts regularly* on Z if every orbit has the same dimension and is regular.

Definition 8.2. Given a subset $S \subset Z$, the *saturation* of S is the union of all the orbits passing through S.

Given a local group of transformations acting on a smooth manifold Z, let Z/G denote the quotient set of all orbits of G, and let $\pi_G\colon Z \to Z/G$ be the projection which associates to each point in Z the orbit of G passing through that point. There is a natural topology on Z/G given by the images of saturated open subsets of Z under the projection π_G. As usual, let $\mathfrak{g}$ denote the involutive differential system spanned by the infinitesimal generators of G.

Theorem 8.3 [19, *p*. 19]. *If G acts regularly on the smooth manifold Z, then the quotient space Z/G can be endowed with the structure of a smooth manifold such that the projection $\pi_G: Z \to Z/G$ is a smooth map, the null space of $d\pi_G|_z$ is $\mathfrak{g}|_z$, and the range is $T(Z/G)|_{\pi_G(z)}$.*

It should be remarked that the quotient manifold Z/G does not necessarily satisfy the Hausdorff topological axiom. For instance, let $Z = \mathbf{R}^2 - \{0\}$, and let G be the one-parameter group generated by the vector field

$$\mathbf{v} = (x^2 + u^2)\partial_x,$$

which is a local regular group action on Z. The quotient manifold Z/G can be realized a copy of the real line with two infinitely close origins, which are given by the orbits $\{(x, 0): x > 0\}$ and $\{(x, 0): x < 0\}$. It is, however, entirely possible to develop a theory of smooth manifolds which does not use the Hausdorff separation axiom, with little change in the relevant results in the local theory. All the results of this paper hold in the non-Hausdorff case; after consulting Palais' monograph [19], the interested reader may check this.

Definition 8.4. A *local G-invariant p-section* of Z is a p-dimensional submanifold $s \subset Z$ such that for each point $z \in s$ there is an open $e \in N_z \subset G_z$ with the property that any transformation $g \in N_z$ satisfies $g \cdot z \in s$. A *global G-invariant p-section* is a p-dimensional submanifold $s \subset Z$ which is invariant under all the transformations in G.

Since G acts regularly, any local G-invariant p-section can be extended to a global G-invariant p-section simply by taking its saturation. Note that a necessary condition for G to admit invariant p-sections is that $p \geqslant l$ and in this case, the global invariant p-sections are in one-to-one correspondence via the projection π_G with the $(p - l)$-dimensional submanifolds of Z/G.

Let us look at the construction of the quotient manifold Z/G from a more classical viewpoint. The regularity of the action of G implies the existence of regular coordinates $\chi: V \to \mathbf{R}^n$, where V is an open subset of Z and for any orbit $\mathcal{O}$ of G

$$\chi(V \cap \mathcal{O}) = \{z = (z^1, \cdots, z^n): z^{l+1} = c_{l+1}, \cdots, z^n = c_n\}$$

for some constants $c_{l+1}, \cdots, c_n$. Now suppose that $z = (z^1, \cdots, z^n)$ is any system of local coordinates on Z. A real valued function $F: Z \to \mathbf{R}$ is called an invariant of G if $F(gz) = F(z)$ for all $z \in Z$, $g \in G_z$. By the existence of regular coordinate systems on Z, we know that locally there always exist $n - l$ functionally independent invariants of G, say $F^1, \cdots, F^{n-l}$. Let $F = (F^1, \cdots, F^{n-l}): Z \to \mathbf{R}^{n-l}$. The functional independence of these invariants is another way of saying that the Jacobian map $dF: TZ \to T\mathbf{R}^{n-l}$ has maximal rank. We conclude that these invariants provide local coordinates on the quotient manifold Z/G. The reader should consult Ovsjannikov [17, Chapter 3] for an exposition of the subject from this viewpoint, although no explicit reference is made to the quotient manifold.

The main goal of this section is to provide an easy characterization of the subbundle of $J_k^*(Z,p)$ given by the extended k-jets of G-invariant p-sections of Z and to show how this subbundle is related to $J_k^*(Z/G, p-l)$. This in turn will yield as a direct corollary the fundamental theorem on the existence of G-invariant solutions to systems of partial differential equations invariant under the prolonged action of G.

Lemma 8.5. *If $s \subset Z$ is a global G-invariant p-section, then its projection $s/G = \pi_G(s) \subset Z/G$ is a $(p-l)$-section of Z/G. Conversely, if $s/G \subset Z/G$ is a $(p-l)$-section of Z/G, then $\pi_G^{-1}(s/G)$ is a global G-invariant p-section of Z. Moreover, $d^k\pi_G$ maps $\mathfrak{T}_k s|_z$ onto $\mathfrak{T}_k(s/G)|_{\pi_G z}$.*

Given a section ω of the k-th order tangent bundle $\mathfrak{T}_k Z$, for each $z \in Z$ there is a well-defined map

$$\cdot \odot \omega|_z : \mathfrak{T}_k Z|_z \to \mathfrak{T}_{k+l} Z|_z$$

depending smoothly on z and given by the formula

$$\nu \odot \omega|_z(f) = \nu\big[\omega(f(z))\big], \quad \nu \in \mathfrak{T}_l Z|_z, \ f \in C^\infty(Z, \mathbf{R}).$$

In future, the dependence of this map on z will be suppressed, so the above formula is more succinctly written $\nu \odot \omega(f) = \nu[\omega(f)]$. If Ω is a k-prolonged differential system, i.e., a vector subbundle of $\mathfrak{T}_k Z$, and Λ a vector subspace of $\mathfrak{T}_l Z|_z$, then $\Lambda \odot \Omega$ will denote the vector subspace of $\mathfrak{T}_{k+l} Z|_z$ spanned by all $\lambda \odot \omega$ for $\lambda \in \Lambda$ and ω a section of Ω. If Ω' is an l-prolonged differential system, then $\Omega \odot \Omega'$ is a $(k+l)$-prolonged differential system. Note that $\Omega \odot \Omega'$ and $\Omega' \odot \Omega$ are not necessarily equal. Define

$$\mathfrak{g}^{(1)} = \mathfrak{g}, \quad \mathfrak{g}^{(k)} = \mathfrak{g} \odot \mathfrak{g}^{(k-1)}.$$

Lemma 8.6. *Suppose $\Lambda \in \mathrm{Grass}^{(k)}(\mathfrak{T}_k Z, p)|_z$ and $\mathfrak{g}^{(k)}|_z \subset \Lambda$. Then there exists a locally G-invariant submanifold $s \subset Z$ with $\mathfrak{T}_k s|_z = \Lambda$.*

Definition 8.7. Define the *G-invariant k-jet subbundle* of p-sections of Z:

$$\mathrm{Inv}^{(k)}(G,p) = \big\{\Lambda \in \mathrm{Grass}^{(k)}(\mathfrak{T}_k Z, p) \colon \Lambda_1 \supset \mathfrak{g}, \Lambda_{i+1} \supset \Lambda_i \odot \mathfrak{g}, \quad i = 1, \cdots, k-1\big\}$$
$$\subset \mathrm{Grass}^{(k)}(\mathfrak{T}_k Z, p) \simeq J_k^*(Z,p).$$

The next theorem shows that $\mathrm{Inv}^{(k)}(G,p)$ actually does represent the subbundle of G-invariant p-sections, and gives its "identification" with the extended k-jet bundle of $(p-l)$-sections of Z/G.

Theorem 8.8. *There is a natural map*

$$\pi_G^{(k)} : \mathrm{Inv}^{(k)}(G, p) \to J_k^*(Z/G, p - l)$$

with the following properties:

(i) $\pi_G^{(k)} : \mathrm{Inv}^{(k)}(G, p)|_z \tilde{\to} J_k^*(Z/G, p - l)|_{\pi_G z}$ *gives an isomorphism of fibers.*

(ii) *Given a p-section* $s \subset Z$, *there exists a* $(p - l)$*-section* $s/G \subset Z/G$ *with* $\pi_G(s) = s/G$ *iff* $j_k^* s \subset \mathrm{Inv}^{(k)}(G, p)$, *in which case* $\pi_G^{(k)}(j_k^* s) = j_k^*(s/G)$.

Proof. First suppose that $j_k^* s \subset \mathrm{Inv}^{(k)}(G, p)$. In particular, this implies that for each $z \in s$, $Ts|_z \supset \mathfrak{g}|_z$, which shows that s is locally G-invariant.

Conversely, suppose s is a G-invariant p-section of Z with image $\pi_G(s) = s/G$. Choose local coordinates $(z^1, \cdots, z^{p+q})$ centered at $z \in s$ so that $\mathfrak{g}$ is spanned by $\{\partial_1, \cdots, \partial_l\}$ and s is given by $\{z: z^{p+1} = \cdots = z^{p+q} = 0\}$. Thus $\mathfrak{T}_i s|_z$ is spanned by $\{\partial_I|_z: \Sigma I \leqslant i,\ i_{p+1} = \cdots = i_{p+q} = 0\}$. Hence $\mathfrak{T}_{i+1} s|_z \supset \mathfrak{T}_i s|_z \odot \mathfrak{g}$ for all i. Finally, given $\Lambda \in \mathrm{Inv}^{(k)}(G, p)|_z$, by Lemma 8.6 there exists a locally G-invariant p-section s of Z with $\mathfrak{T}_k s|_z = \Lambda$. Define $\pi_G^{(k)}(\Lambda) = d^k \pi_G(\Lambda)$, so that by Lemma 8.5, $\pi_G^{(k)}(j_k^* s|_z) = j_k^*(s/G)|_{\pi_G z}$.

Theorem 8.9. *Let* Δ_0 *be a k-th order system of partial differential equations on Z. Let* $\Delta_0' = \Delta_0 \cap \mathrm{Inv}^{(k)}(G, p)$ *be the corresponding system of partial differential equations for G-invariant solutions to* Δ_0. *If* Δ_0' *is invariant under the prolonged group action* $\mathbf{pr}^{(k)}G$, *then a p-section s of Z is a G-invariant solution to* Δ_0 *iff* s/G *is a solution to the reduced differential equation* $\Delta_0/G = \pi_G^{(k)}(\Delta_0')$. *In particular, if* Δ_0 *is G-invariant, then* Δ_0' *is also G-invariant.*

Proof. If s is a G-invariant solution to Δ_0, then $j_k^* s \subset \Delta_0 \cap \mathrm{Inv}^{(k)}(G, p)$ by the previous theorem, and therefore $\pi_G^{(k)}(j_k^* s) = j_k^*(s/G) \subset \Delta_0/G$. Conversely, given s/G, a solution to Δ_0/G, let s be the corresponding G-invariant p-section of Z. By the isomorphism given in (i) of Theorem 8.8, $j_k^*(s/G)|_{\pi_G z} \in \Delta_0/G|_{\pi_G z}$ implies $j_k^* s|_z \in \Delta_0'|_z \subset \Delta_0|_z$, giving the result. Finally, the last statement of the theorem follows from the fact that $\mathrm{Inv}^{(k)}(G, p)$ is a $\mathbf{pr}^{(k)}G$ invariant subbundle of $J_k^*(Z, p)$.

To show that this theorem is the optimal result on the existence of a system of partial differential equations Δ_0/G on Z/G whose solutions give the G-invariant solutions to Δ_0, we briefly consider a few elementary examples. On the manifold $Z = \mathbf{R}^3$ with coordinates (x, y, u) consider the first order equation $\Delta_0 = \{x u_x + u_y = 0\}$. Let G be the one-parameter group of translations in the x-coordinate. The equation Δ_0 is not G-invariant, but $\Delta_0' = \{u_x = 0, u_y = 0\}$ is and admits the G-invariant solutions u = constant. With G and Z the same, consider the equation $\Delta_0 = \{u_y - xu - x^2 u_x = 0\}$, which is again not G-invariant. Δ_0' is also not G-invariant, but it admits the solution $u = 0$. In this case $\Delta_0'|\{u = 0\}$ is G-invariant. However, even this is not necessarily true as the example $\Delta_0 = \{u_y - xu = xu_z\}$ on $\mathbf{R}^4$ with G being translation in the x-coordinate shows. Again $u = 0$ is a G-invariant solution to Δ_0, but $\Delta_0'|\{u = 0\}$ is not G-invariant.

For an example of a case when Δ_0 is not G-invariant, but Δ_0' is and interesting solutions are obtained, see Bluman and Cole [3].

It remains to demonstrate how the reduced equation Δ_0/G of Theorem 8.9 is found in practice. Let $(\xi, \zeta) = (\xi^1, \cdots, \xi^p, \zeta^1, \cdots, \zeta^q)$ be local coordinates on Z/G. Let $(x, u) = (x^1, \cdots, x^p, u^1, \cdots, u^q)$ be local coordinates on Z. The projection π_G is described by the equation

$$\pi_G(x, u) = (\bar{\xi}(x, u), \bar{\zeta}(x, u)),$$

where the $\bar{\xi}^i$'s and $\bar{\zeta}^j$'s form a complete set of functionally independent invariants of G. Assume for convenience that the orbits of G are all transversal to the fibers $x =$ constant. Since $p \geqslant l$, the transversality condition allows us to find a smooth function $\omega(x, \xi, \zeta)$ such that the orbits of G are just

$$\pi_G^{-1}(\xi, \zeta) = \{(x, u) : u = \omega(x, \xi, \zeta)\}, (\xi, \zeta) \in Z/G.$$

Suppose $s = \{u = f(x)\}$ is a G-invariant p-section of Z with $s/G = \{\zeta = \phi(\xi)\}$, where we have assumed that s/G is transversal to the fibers $\xi =$ constant. Therefore

$$f(x) = \omega\big(x, \bar{\xi}(x, f(x)), \phi(\bar{\xi}(x, f(x)))\big).$$

Differentiating this equation with respect to x yields

$$\partial_x f(x) = \partial_x \omega + \left[\partial_\xi \omega + \partial_\zeta \omega \cdot \partial_\xi \phi\right] \cdot \left(\partial_x \bar{\xi} + \partial_u \bar{\xi} \cdot \partial_x f\right),$$

where the subscripts on the ∂'s indicate with respect to which set of variables the partial derivatives are being taken. Again the transversality conditions allow us to solve for ∂f:

$$\partial f = \omega^{(1)}\big(x, \bar{\xi}, \phi(\bar{\xi}), \partial\phi(\bar{\xi})\big).$$

Continuing to differentiate, we compute

$$\partial_*^k f = \omega^{(k)}\big(x, \bar{\xi}, \phi(\bar{\xi}), \ \partial_*^k \phi(\bar{\xi})\big), \quad k = 1, 2, \cdots. \tag{8.1}$$

We conclude from part (ii) of Theorem 8.8 that

$$\left[\pi_G^{(k)}\right]^{-1}(\xi, \zeta, \zeta^{(k)}) = \left\{(x, u, u^{(k)}) : u = \omega(x, \xi, \zeta), u^{(k)} = \omega^{(k)}(x, \xi, \zeta, \zeta^{(k)})\right\},$$
$$(\xi, \zeta, \zeta^{(k)}) \in J_k^*(Z/G, p - l).$$

Now suppose that $\Delta_0 \subset J_k^*(Z, p)$ is a system of partial differential equations which is invariant under the prolongation of the action of G, and that in local coordinates the subvariety Δ_0 is described by the equations

$$\Delta^i(x, u, u^{(k)}) = 0, \ i = 1, \cdots, \alpha.$$

The equation Δ_0/G is found by substituting the expressions (8.1) for the derivatives of u, giving

$$\tilde{\Delta}^i(x, u, \xi, \zeta, \zeta^{(k)}) = 0, \ i = 1, \cdots, \alpha.$$

Now Theorem 8.9 assures us that these equations are equivalent to a system depending solely on the variables of the quotient manifold:

$$\hat{\Delta}^i(\xi, \zeta, \zeta^{(k)}) = 0, \ i = 1, \cdots, \alpha.$$

This final system is the reduced equation Δ_0/G. This process will become clearer in the following example.

Example 8.10 (*Burgers' equation*). Consider the equation

$$\Delta_0 : u_t + uu_x + u_{xx} = 0,$$

whose symmetry group was calculated in Example 6.1.

Various one-parameter subgroups of this group will be considered, and the invariant solutions corresponding to them will be derived. As a first example, the traveling wave solutions will be found. These correspond to the vector field $c\partial_x + \partial_t$, where c is the wave velocity. This exponentiates to the group action

$$G_c : (x, t, u) \mapsto (x + \lambda c, t + \lambda, u), \ \lambda \in \mathbf{R},$$

which has independent invariants u and $\xi = x - ct$. We have

$$u_t = -cu', \ u_x = u', \ u_{xx} = u'',$$

where the primes mean derivatives with respect to ξ. The equation Δ_0/G_c on $\mathbf{R}^3/G_c \simeq \mathbf{R}^2$ for the G_c-invariant solutions is then

$$u'' + (u + c)u' = 0.$$

This has a first integral

$$u' + \tfrac{1}{2}u^2 + cu + k_0 = 0.$$

Let $d = 2k_0 - c^2$. Then the G_c invariant solutions are

$$u(x, t) = \begin{cases} \sqrt{d} \tan\left[\tfrac{1}{2}\sqrt{d}\,(ct - x + \delta)\right] - c, & d > 0, \\ 2(x - ct + \delta)^{-1} - c, & d = 0, \\ \sqrt{d} \tanh\left[\tfrac{1}{2}\sqrt{d}\,(x - ct + \delta)\right] - c, & d < 0, \end{cases}$$

where δ is a constant.

Next consider the vector field $\mathbf{v}_4 = t\partial_x + \partial_u$ whose one-parameter group is given in (6.9c). Coordinates on $\mathbf{R}^3/G_4 \simeq \mathbf{R}^2$ are given by the invariants t and $\zeta = u - x/t$. Then

$$u_t = \zeta' - x/t^2, \ u_x = 1/t, \ u_{xx} = 0.$$

The equation Δ_0/G_4 is just

$$t\zeta' + \zeta = 0,$$

which is of first order. Therefore the general G_4-invariant solution to Δ_0 is

$$u(x, t) = (x + k_0)/t,$$

for some constant k_0. Similarly it can be shown that the invariant solutions for the infinitesimal operator $(a + t)\partial_x + \partial_u$ are

$$u(x, t) = (x + k_0)/(t + a).$$

A more interesting case arises when the scale invariant solutions are considered. The vector field here is $\mathbf{v}_3 = x\partial_x + 2t\partial_t - u\partial_u$, corresponding to the group G_3 given in (6.9b). To let G_3 act regularly, we must restrict our attention to the submanifold $Z' = \mathbf{R}^3 - \{0\}$. The quotient manifold Z'/G_3 is non-Hausdorff. It can be realized as a cylinder $S^1 \times \mathbf{R}$ together with two exceptional points L_+ and L_-, which correspond to the two vertical orbits

$$L_+ = \{x = t = 0, u > 0\}, \ L_- = \{x = t = 0, u < 0\}.$$

If (θ, h) are the coordinates on $S^1 \times \mathbf{R}$, then neighborhood bases of L_+ and L_- are given by

$$\{L_+\} \cup \{(\theta, h) : 0 < h < \varepsilon\}, \ \{L_-\} \cup \{(\theta, h) : -\varepsilon < h < 0\}, \ \varepsilon > 0,$$

respectively. A G_3-invariant solution of Burgers' equation corresponds to a curve in Z'/G_3 which is a solution to Δ_0/G_3. Note that if the curve passes through either of the exceptional points, the corresponding G_3-invariant solution is not a single valued function of (x, t), so we will concentrate on the Hausdorff submanifold $S^1 \times \mathbf{R} \subset Z'/G_3$. Using the local coordinates

$$\xi = t^{-1}x^2, \ w = xu,$$

and treating ξ as the new independent variable, we see that

$$\begin{aligned} u_t &= -xt^{-2}w', \\ u_x &= -x^{-2}w + 2t^{-1}w', \\ u_{xx} &= 2x^{-3}w - 2t^{-1}x^{-1}w' + 4t^{-2}xw''. \end{aligned}$$

In these coordinates, the equation Δ_0/G_3 is

$$4\xi^2 w'' + \xi(2w - 2 - \xi)w' + w(2 - w) = 0.$$

If $w = 4\xi\phi'/\phi$ where ϕ is a smooth positive function of ξ, then the above equation reduces to

$$4\xi^2\left[\frac{4\xi\phi'' - \xi\phi' + 2\phi'}{\phi}\right]' = 0,$$

or, upon integration,

$$4\xi\phi'' + (2 - \xi)\phi' - k\phi = 0$$

for some constant k. This is (up to multiplication by a constant) the confluent hypergeometric equation and has general solution

$$\phi(\xi) = c \cdot {}_1F_1(k, 1/2; \xi/4) + c'\sqrt{\xi} \cdot {}_1F_1(k + 1/2, 3/2; \xi/4).$$

(See for instance [27, Chapter 6].) From any particular ϕ we may reconstruct local G_3-invariant solutions of Burgers' equation via the formula

$$u(x, t) = \frac{4x}{t}\frac{\phi'(x^2/t)}{\phi(x^2/t)}.$$

Finally, consider the vector field $\mathbf{v}_5 = xt\partial_x + t^2\partial_t + (x - tu)\partial_u$ with one-parameter group G_5 which acts regularly on $Z'' = \mathbf{R}^3 \sim \{x = t = 0\}$. Then $Z''/G_5 \simeq S^1 \times \mathbf{R}$. (A cross-section to the orbits of G_5 is provided by a line bundle with two twists over the circle $\{(x, t, u)\colon u = 0, x^2 + t^2 = 1\}$.) Convenient local coordinates are given by the invariants

$$\xi = \frac{x}{t}, \quad w = tu - x.$$

The reduced equation Δ_0/G_5 is

$$w'' + ww' = 0,$$

which gives the G_5 invariant solutions

$$u(x, t) = \frac{-k}{t}\left[\tan\left(\frac{kx + k't}{2t}\right) + x\right], \quad u(x, t) = \frac{x + k}{t},$$

where k and k' are arbitrary real constants. Other G_5-invariant solutions can be found by using different coordinate patches on Z''/G_5.

Symbol index

symbol		*page*
$\odot_k V$	k-th symmetric power of a vector space V	501
$\odot_k^* V$	$= \bigoplus_{i=1}^{k} \odot_i V$	503
$\odot_* V$	symmetric algebra of a vector space V	501
$\odot^k(V, W)$	space of W-valued k-symmetric linear functions on V	501
$\odot_*^k(V, W)$	$= \bigoplus_{i=1}^{k} \odot^i(V, W)$	503
$\odot^*(V, W)$	space of symmetric W-valued linear functions on V	501
$\odot$	(i) product in $\odot_* V$	501
	(ii) product in $\odot^*(V, W)$ when W is an algebra	502
$\mathfrak{T}_k M$	k-th order tangent bundle of manifold M	507

$\mathcal{T}_k^* M$	k-th order cotangent bundle of manifold M	507
ΣI	rank of multi-index I	501
$\mathbb{S}_k^n$	set of n-multi-indices of rank k	501
$\mathbb{S}^n$	set of all n-multi-indices	501
δ^σ	Kronecker multi-index–$(0, \cdots, 0, \overset{\sigma}{1}, 0, \cdots, 0)$	501
$I!$	Factorial of multi-index I	501
$\binom{I}{J}$	Binomial coefficient for multi-indices I, J	501
$\mathcal{Q}_k$	k-Faa-di-Bruno set of multi-indices	503
$\mathcal{Q}_k^j$	k-Faa-di-Bruno set of multi-indices of rank j	505
$N_{k,p}$	dimension of $\odot_k^* \mathbf{R}^p$	510
∂_k	partial derivative in x^k direction	502
∂_I	partial derivative corresponding to multi-index I	502
$\partial^k f$	k-th order differential of map f between vector spaces	502
$\partial_*^k f$	$= \partial f + \partial^2 f + \cdots + \partial^k f$	504
ε_k	Faa-di-Bruno injection	503
π_k	projection inverse to ε_k	503
$d^k f$	(i) $\varepsilon_k[\partial_*^k f]$	504
	(ii) induced map on k-th order tangent bundles	507
$\partial_j^i f$	matrix blocks of $d^k f$	505
D_k	total derivative in x^k direction	506
D_I	total derivative corresponding to multi-index I	505
$D^k f$	k-th order total differential of function f	505
$D_*^k f$	$= Df + D^2 f + \cdots + D^k f$	505
$\mathbf{1}_X$	identity map of a space X	
$\mathbf{1}_p$	identity map of $\mathbf{R}^p$	
$GL^{(k)}(V)$	k-th order prolonged general linear group of V	504
$GL^{(k)}(n)$	k-th order prolonged linear group of $\mathbf{R}^n$	504
$PGL^{(k)}(n)$	k-th order prolonged projective linear group of $\mathbf{R}^n$	513
$\mathrm{Grass}^{(k)}(V, p)$	k-th order prolonged Grassmannian of prolonged p-planes for vector space V	509
$\mathrm{Grass}^{(k)}(V, p; W)$	trivialized prolonged Grassmannian	512
$C^\infty(M, p)\|_m$	space of germs of smooth p-sections of M passing through m	506
$J_k^*(M, p)$	k-th order extended jet bundle of p-sections of M	508
$J_k^*(M, p; \mathcal{U})$	trivialized k-th order extended jet bundle	513
$j_k^* s$	k-th order extended jet of p-section	508
π_j^k	(i) canonical prolonged Grassmannian projection	509
	(ii) canonical extended jet bundle projection	513
ι_j^k	canonical extended jet bundle injection	516
$\mathrm{pr}^{(k)}\Delta$	k-th prolongation of differential operator Δ	515

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